The Navier wall law at a boundary with random roughness

David Gérard-Varet *

Abstract

We consider the Navier-Stokes equation in a domain with irregular boundaries. The irregularity is modeled by a spatially homogeneous random process, with typical size $\varepsilon \ll 1$. In the parent paper [8], we derived a homogenized boundary condition of Navier type as $\varepsilon \to 0$. We show here that for a large class of boundaries, this Navier condition provides a $O(\varepsilon^{3/2}|\ln \varepsilon|^{1/2})$ approximation in L^2, instead of $O(\varepsilon^{3/2})$ for periodic irregularities. Our result relies on the study of an auxiliary boundary layer system. Decay properties of this boundary layer are deduced from a central limit theorem for dependent variables.

Keywords: Wall laws, rough boundaries, stochastic homogenization, decay of correlations

1 Introduction

The concern of this paper is the effect of a rough boundary on a viscous fluid. In most situations of physical relevance, such effect can not be described in detail: either the precise shape of the roughness is unknown, or its spatial variations are too small for computational grids. Therefore, one may only hope to account for the averaged effect of the irregularities. This is the purpose of *wall laws*: the irregular boundary is replaced by an artificial smoothed one, and an artificial boundary condition (a wall law) is prescribed there, that should reflect the mean impact of the roughness.

This paper is a mathematical study of wall laws, in the following simple setting: we consider a two-dimensional rough channel

$$\Omega^\varepsilon \;=\; \Omega \cup \Sigma \cup R^\varepsilon$$

where $\Omega = \mathbb{R} \times (0,1)$ is the *smooth part*, R^ε is the rough part, and $\Sigma = \mathbb{R} \times \{0\}$ their interface. We assume that the rough part has typical size ε, that is

$$R^\varepsilon \;=\; \left\{ x,\; x_2 > \varepsilon\omega\left(\frac{x_1}{\varepsilon}\right)\right\}$$

for a $K-$Lipschitz function $\omega : \mathbb{R} \mapsto (-1,0)$, $K > 0$. More will be assumed on the boundary function ω hereafter (see figure for an example of such a rough domain).

*DMA/CNRS, Ecole Normale Supérieure, 45 rue d'Ulm,75005 Paris, FRANCE

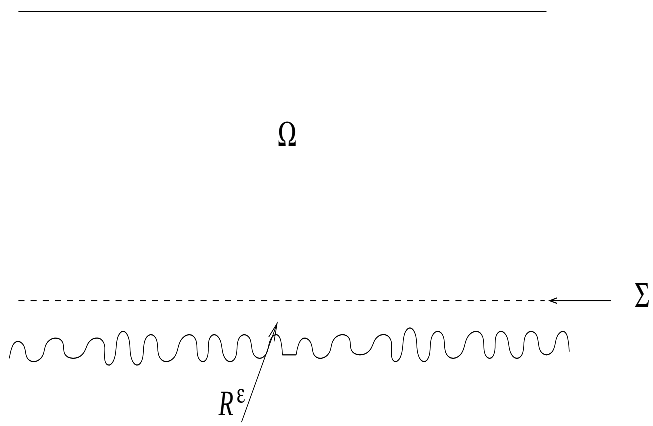

Figure 1: The rough domain Ω^ε.

We assume that in this channel domain, the viscous fluid obeys to the stationary incompressible Navier-Stokes equations:

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = 0, \; x \in \Omega^\varepsilon, \\ \text{div } u = 0, \; x \in \Omega^\varepsilon, \\ \displaystyle\int_{\sigma^\varepsilon} u_1 = \phi, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where σ^ε denotes any vertical cross-section of Ω^ε and $\phi > 0$. The third equation in (1.1) expresses that a flux ϕ is imposed across the channel. Note that this flux does not depend on the cross-section, due to the incompressibility and no-slip condition at the boundary. We also stress that, up to minor changes, we could apply our analysis to many variants of this problem, notably to elliptic type systems or to unstationary Navier-Stokes.

In this simple setting, the search for wall laws resumes to the following problem: to find a boundary operator $B^\varepsilon(x, D_x)$, regular in ε, acting at the *artificial* boundary Σ, such that the solution of

$$\begin{cases} -\Delta u + u \cdot \nabla u + \nabla p = 0, \; x \in \Omega, \\ \text{div } u = 0, \; x \in \Omega, \\ \displaystyle\int_{\sigma} u_1 = \phi, \quad u|_{x_2=1} = 0, \\ B^\varepsilon(x, D_x) u|_\Sigma = 0 \end{cases} \tag{1.2}$$

approximates well the solution u^ε of (1.1) in Ω.

This type of homogenization problems has been considered in many mathematical works. On wall laws for scalar elliptic equations, we refer to [2]. On wall laws for fluid flows, see [1, 3, 4, 19, 20, 11]. See also [21] on porous boundaries. These works go along with more formal computations, grounded by empirical arguments (*cf* for instance [9, 23]). We finally mention [15, 10] for the study of roughness-induced effects on geophysical systems.

All these studies have been carried under two assumptions:

- compact domains, for instance bounded channels or periodic in the variable x_1.
- *periodic irregularities*, meaning that the boundary function ω is periodic.

The first restriction is just a small mathematical convenience, that gives direct compactness properties through Rellich type theorems. The second assumption is of course a big simplification, both from the point of view of mathematics and physics. These assumptions were considerably relaxed in the recent article [8] by A. Basson and the author. As the present note extends this article, we now describe shortly its main results and underlying difficulties.

In all papers on wall laws, the starting point is a formal expansion of u^ε:

$$u^\varepsilon(x) \;\sim\; u^0(x) + 6\phi\varepsilon v(x/\varepsilon) + \dots$$

Formally, the leading term u^0 satisfies (1.2) with the simple no-slip condition

$$B^\varepsilon(x, Dx)u \;:=\; u = 0 \;\text{ at }\; \Sigma \tag{1.3}$$

The solution of this approximate system is the famous Poiseuille flow :

$$u^0(x) \;=\; (U(x_2), 0)\,, \quad U(x_2) \;=\; 6\phi x_2(1 - x_2)$$

Note that u^0 is defined in all $\mathbb{R}^2$. This zeroth order asymptotics can be mathematically justified, at least for small fluxes ϕ: we prove in article [8]

Theorem 1 *There exists ϕ_0, $\varepsilon_0 > 0$, such that for all $\phi < \phi_0$, $\varepsilon < \varepsilon_0$, system* (1.1) *has a unique solution u^ε in $H^1_{uloc}(\Omega^\varepsilon)$. Moreover,*

$$\|u^\varepsilon - u^0\|_{H^1_{uloc}(\Omega^\varepsilon)} \le C\sqrt{\varepsilon}, \quad \|u^\varepsilon - u^0\|_{L^2_{uloc}(\Omega)} \le C'\varepsilon.$$

We stress that these estimates hold without further assumption on the boundary: we only assume that ω has values in $(-1, 0)$ and is $K-$Lip. A look at the proof shows that the constants C and C' are only increasing functions of K.

Theorem 1 expresses that the wall law (1.3) provides a $O(\varepsilon)$ approximation of u^ε in $L^2_{uloc}(\Omega)$. See also [19] for a similar result in a bounded channel. However, this wall law does not account for the behaviour of u^ε near the boundary, and can therefore be refined. Indeed, as the Poiseuille flow u^0 does not vanish at the lower part of $\partial\Omega^\varepsilon$, a boundary layer corrector $\varepsilon\phi v(x/\varepsilon)$ must be added to the expansion. The (normalized) boundary layer $v = v(y)$ is defined on the rescaled infinite domain

$$\Omega^{bl} \;=\; \{y,\; y_2 > \omega(y_1)\}$$

and formally satisfies the following Stokes problem

$$\begin{cases} -\Delta v + \nabla q = 0,\; x \in \Omega^{bl}, \\ \operatorname{div} v = 0,\; x \in \Omega^{bl}, \\ v(y_1, \omega(y_1)) \;=\; -(\omega(y_1), 0). \end{cases} \tag{1.4}$$

Note the inhomogeneous Dirichlet condition, that cancels the trace of u^0.

Although linear, the boundary layer system (1.4) *is quite challenging.* First, the well-posedness is not clear. As the boundary function ω is not decreasing at infinity, one can

expect only local integrability of the solution v in variable y_1. The derivation of local bounds is not obvious: the Stokes operator being vectorial, one can not use scalar tools such as the maximum principle or Harnack inequality. Moreover, as Ω^{bl} is unbounded in all directions, the Poincaré inequality (which allows to get H^1_{uloc} estimates in the channel) is not available. Besides the well-posedness issue, the qualitative properties of v seem also out of reach without further hypothesis.

Under an assumption of periodic irregularities, the analysis of (1.4) becomes straightforward. If ω is say L periodic in y_1, it is easy to show well-posedness in the space

$$\left\{ v \in H^1_{loc}(\Omega_{bl}),\ v\ L-\text{periodic in } y_1,\ \int_0^L \int_{\omega(y_1)}^{+\infty} |\nabla v|^2 dy_2 dy_1 < +\infty \right\}.$$

Moreover, a simple Fourier transform in y_1 shows that

$$\|v(y) - v^\infty\| \le C\, e^{-\delta y_2/L}, \quad v^\infty = (\alpha, 0), \quad \alpha = \frac{1}{L}\int_0^L v_1(s) ds, \quad \delta > 0,$$

that is exponential convergence to a constant field $v^\infty = (\alpha, 0)$ at infinity.

The constant α at infinity is then responsible for a $O(\varepsilon)$ tangential slip. Namely, chosing as a wall law the Navier-slip condition

$$B^\varepsilon(x, D_x)v \ = \ (v_1 - \varepsilon\alpha\partial_2 v_1,\ v_2) = 0 \ \text{ at } \Sigma, \tag{1.5}$$

it can be shown (in this periodic framework) that the solution of (1.2) provides a $O(\varepsilon^{3/2})$ approximation of u^ε in L^2. We refer to [19] for all necessary details. The error estimate $\varepsilon^{3/2}$ comes from the fact that the boundary layer term satisfies $\|\varepsilon(v(x/\varepsilon) - (\alpha, 0))\|_{L^2} = O(\varepsilon^{3/2})$.

The periodicity hypothesis is a stringent one, and one may wonder if the use of Navier slip condition can be justified in more general configurations. This issue has been adressed rigorously in the recent article [8]. Inspired by the probabilistic modeling of heterogeneous media (see for instance [22]), we considered irregularities that are not distributed periodically, but randomly, following a stationary stochastic process. Namely, the rough boundary is seen as a realization of a stationary spatial process. Following the well-known construction of Kolmogorov, this amounts to consider the space

$$P \ = \ \{\omega : \mathbb{R} \mapsto (-1, 0),\ \omega\ K - \text{Lip}\}$$

of all possible rough boundaries, together with the cylindrical $\sigma-$ field $\mathcal{C}$ (that is generated by the coordinates $\omega \mapsto \omega(t)$) and with a stationary measure π. Stationary means that π is invariant by the group of translation

$$\tau_h : P \mapsto P, \quad \omega \mapsto \omega(\cdot + h).$$

As a consequence of this modeling, the domains Ω^ε, Ω^{bl}, as well as the velocity fields u^ε or v depend on the parameter ω. As discussed earlier, the existence result and estimates of theorem 1 are uniform on P. Moreover, it was shown in article [8] that the function $\omega \mapsto u^\varepsilon(\omega, \cdot)$ (extended by 0 outside $\Omega^\varepsilon(\omega)$) is measurable as a function from P to $H^1_{loc}(\mathbb{R}^2)$.

Using this probabilistic structure, we have been able to extend partially the results of the periodic case. Key elements of our analysis are:

- the well-posedness of the boundary layer system, obtained in functional spaces encoding the relation
$$v(\tau_h(\omega), y_1, y_2) \; = \; v(\omega, y_1 + h, y_2).$$
- the convergence of $v(\omega, y)$ to $(\alpha(\omega), 0)$ as $y_2 \to +\infty$, both in $L^2(P)$ and almost surely, locally uniformly in y_1. Such convergence is deduced from the ergodic theorem.

More on the boundary layer system will be provided in the next sections. As regards the Navier wall law (1.5), the main result of [8] resumes to

Theorem 2 *There exists $\alpha = \alpha(\omega) \in L^2(P)$ such that the solution u^N of* (1.2), (1.5) *satisfies*
$$\|u^\varepsilon - u^N\|_{L^2_{uloc}(P\times\Omega)} = o(\varepsilon).$$

We remind that $\|w\|_{L^2_{uloc}(P\times\Omega)} \; := \; sup_x \left(\int_P \int_{B(x,1)\cap\Omega} |w|^2 dx dP\right)^{1/2}.$

Theorem 2 shows that a slip condition of Navier type improves the approximation of u^ε. As in the periodic case, the random variable α in (1.5) comes from the convergence of the boundary layer v. If the measure π is ergodic, α does not depend on ω, as pointed out in [8].

A natural concern about this result is the $o(\varepsilon)$ bound, which is only a slight improvement of the $O(\varepsilon)$ in theorem 1. A look at article [8] shows that this poor bound is due to the lack of information on the way v converges at infinity. Contrary to the periodic case, where convergence at exponential rate is established, the simple use of the ergodic theorem does not yield any speed rate.

The present paper aims at clarifying this point. Losely, *we will show that for a large class of boundaries, the Navier wall law provides a $O(\varepsilon^{3/2}|\ln(\varepsilon)|^{1/2})$ approximation of the real solution.* Namely, we will make the two following assumptions on our random roughness:
(H1) The measure π is supported by
$$P_\alpha = \{\omega : \mathbb{R} \mapsto (-1, 0), \; \|\omega\|_{C^{2,\alpha}} \; \leq K_\alpha\}$$
for some $\alpha > 0$ and some $K_\alpha > 0$.
(H2) The randon boundary has no correlation at large distances, that is the σ-fields
$$\sigma\left(s \mapsto \omega(s), \; s \leq a\right) \quad and \quad \sigma\left(s \mapsto \omega(s), \; s \geq b\right)$$
are independent for $b - a \geq \kappa$, for some $\kappa > 0$.
Under these assumptions, the main theorem of the paper reads:

Theorem 3 *For small enough ϕ and under (H1)-(H2), the following refined estimate holds:*
$$\|u^\varepsilon - u^N\|_{L^2_{uloc}(P\times\Omega)} = O(\varepsilon^{3/2}|\ln(\varepsilon)|^{1/2}).$$

Before entering the proof of this theorem, let us give a few hints. Theorem 3 is deduced from a central limit theorem for the quantity $v(\omega, y) - (\alpha, 0)$. Broadly, this theorem comes from good properties of the random variables
$$X^n(\omega) \; = \; \int_n^{n+1} v(\omega, y_1, 0)\, dy_1.$$

Due to the elliptic nature of the Stokes operator, such random variables are not independent. However, under assumption (H2), we are able to prove that the correlation terms $E(X_n\, X_0)$ decay fast enough as $n \to \infty$. As a result, one can prove a central limit theorem on X_n, and then a similar one on $v-(\alpha, 0)$. We point out that such type of results for dependent variables with strong decay of correlations is quite classical and has been used in various fields. We refer to [7] for a review paper related to dynamical systems, and to recent articles [24, 12] for applications in a PDE context.

As a consequence of this central limit theorem, we show that the boundary layer converges to a constant as $|y_2^{-1/2}|$. Note that this is in sharp contrast with the periodic case, where exponential convergence holds (we stress that periodic boundaries are highly correlated, thus far from satisfying (H2)).This speed of convergence is resposible for the $\varepsilon^{3/2} |\ln(\varepsilon)|^{1/2}$ in the Navier wall law.

The main difficulty is to obtain the decay of correlations of variables like X_n. The proof relies on precise estimates of the Green function for the Stokes operator above a non flat boundary. Such estimates follow from sharp elliptic regularity results, where one must pay attention to the oscillation of the boundary. This is achieved under the regularity assumption (H1), using ideas of Avellaneda and Lin for homogenization of elliptic systems [5, 6].

2 Boundary layer decay and Navier approximation

In this section, we explain how Theorem 2 follows from estimates on the solution v of (1.4). Such estimates will be established in the following sections. At first, we remind the main features of v, as stated in article [8].

2.1 The boundary layer system

As emphasized in the introduction, to solve (1.4) in a deterministic way, that is for each possible boundary ω, is still unclear. Hence, one must take advantage of the probabilistic setting. First, notice that a reasonable solution v should satisfy:

$$v(\tau_h(\omega), y_1, y_2) \;=\; v(\omega, y_1 + h, y_2). \tag{2.1}$$

Together with the stationarity assumption, this relation sort of substitutes to the identity

$$v(y_1 + L, y_2) \;=\; v(y_1, y_2)$$

used in the treatment of $L-$periodic roughness. It allows to extend the well-posedness result, through an appropriate variational formulation.

This formulation has been described in article [8]. First, one introduces the new unknown

$$w(y) \;:=\; v(y) \;+\; (y_2, 0)\, \mathbf{1}_{\{y_2<0\}}(y),$$

and replace system (1.4) by

$$\begin{cases} -\Delta w + \nabla q = 0, \; x \in \Omega^{bl} \setminus \{y_2 = 0\}, \\ \mathrm{div}\; w = 0, \; x \in \Omega^{bl}, \\ w|_{\partial\Omega^{bl}} \;=\; 0, \\ [w]|_{y_2=0} = 0, \quad [\partial_2 w - (0, q)]|_{y_2=0} = (-1, 0), \end{cases} \tag{2.2}$$

where $[\cdot]|_{y_2=0}$ denotes the jump at $y_2 = 0$. Then, one multiplies formally the Stokes equation by a test function $w' = w'(\omega, y)$ that satisfies div $w' = 0$, $w'|_{\partial\Omega^{bl}} = 0$. Integrating by parts over $\Omega^{bl} \cap \{|y_1 < 1\}$ yields

$$\int_{\Omega^{bl}\cap\{|y_1<1\}} \nabla w \cdot \nabla w' \; = \; \int_{\{|y_1|<1,\, y_2=0\}} w'_1 \; + \int_{\Omega^{bl}\cap\{|y_1|=1\}} (\partial_n w - qn)\, w'.$$

Finally, if w, w' satisfy relation (2.1), one can integrate with respect to ω, and thanks to the stationarity of π, get rid of the annoying boundary term at the r.h.s:

$$\mathbb{E} \int_{\Omega^{bl}\cap\{|y_1<1\}} \nabla w \cdot \nabla w' \; = \; \mathbb{E} \int_{\{|y_1<1,\, y_2=0\}} w'_1$$

Afterwards, this formal variational formulation can be rigorously defined and solved: in short, one can apply the Riesz theorem in a functional space of Sobolev type, made of functions w such that

$$\mathbb{E} \int_{\Omega^{bl}\cap\{|y_1<1\}} |\nabla w|^2 \; < \; +\infty,$$

and satisfying almost surely (2.1), together with div $w = 0$, $w|_{\Omega^{bl}} = 0$. We refer to [8] for all details. Note that stationarity implies:

$$sup_{t,R}\, \mathbb{E}\, \frac{1}{R} \int_{\Omega^{bl}\cap\{|y_1-t|<R\}} |\nabla w|^2 \; = \; \mathbb{E} \int_{\Omega^{bl}\cap\{|y_1<1\}} |\nabla w|^2 \; < \; +\infty.$$

Back to the original system (1.4), this variational solution w provides almost surely a solution $v(\omega, \cdot) \in H^1_{loc}\left(\Omega^{bl}\right)$ in the sense of distributions. Moreover, the ergodic theorem yields (see [8])

$$\sup_R \frac{1}{R} \int_{\Omega^{bl}\cap\{|y_1|<R\}} |\nabla v|^2 \; < \; +\infty, \text{ almost surely.}$$

In order to understand the origin of the Navier approximation, the next step is to describe the behavior of v as $y_2 \to +\infty$. For periodic roughness, one can show exponential convergence of v to a constant vector field $v^\infty = (\alpha, 0)$. However, the rate of convergence goes to zero with the period L. When dealing with stationary random boundaries, that broadly speaking contain all periods, the exponential decay does not hold *a priori*. In other words, there is a problem associated to the Fourier spectrum, that is discrete in the periodic case, and may accumulate to zero in the random case.

Again, this problem has been (partially) overcome in [8]. The first step is to obtain a representation of v in terms of a Stokes double layer potential, *cf* proposition ??. Almost surely, for any

$$\begin{aligned} v(\omega, y) \; &= \; \int_{\mathbb{R}} G(t, y_2)\, v(\omega, y_1 - t, 0)\, dt \\ &= \; -\int_{\mathbb{R}} t\, \partial_t G(y_2) \frac{1}{t} \int_0^t v(\omega, y_1 - s, 0)\, ds\, dt \end{aligned}$$

where G is the Poisson type kernel for the Stokes operator over a half space. Then, the ergodic theorem and a few calculations yield:

$$\frac{1}{t} \int_0^t v(\omega, y_1 - s, 0)\, ds \;\; \to v^\infty(\omega) \; = \; (\alpha(\omega), 0), \quad t \to \pm\infty,$$

where the convergence holds almost surely (locally uniformly in y_1), as well as in $L^2(P)$ (uniformly in y_1). In the case where the stationary measure π is ergodic, the constant α does not depend on ω. Finally, back to the integral representation, and with similar treatment for derivatives of v:

$$\forall \beta \in \mathbb{N}^2,\ |\beta| \geq 1, \quad \mathbb{E}\,|v(\cdot,0,,y_2) - \alpha(\cdot)|^2 \;+\; y_2^{2|\beta|}\mathbb{E}\left|\partial_y^\beta\, v(\cdot,0,y_2)\right|^2 \xrightarrow[y_2\to+\infty]{} 0. \tag{2.3}$$

We refer to [8, Proposition 13] for all details.

2.2 Refined estimate for Navier wall law

Most of the analysis of the present paper will be devoted to a refined asymptotic estimate of the boundary layer:

Theorem 4 *Under assumptions (H1), (H2), for all $\beta \in \mathbb{N}^2$,*

$$y_2^{2|\beta|+1}\mathbb{E}\left|\partial^\beta\left(v(\cdot,0,y_2) - \alpha(\cdot)\right)\right|^2 \xrightarrow[y_2\to+\infty]{} \sigma_\beta \geq 0. \tag{2.4}$$

It is of course a much sharper convergence result than (2.3). Before tackling its proof, we explain how it implies theorem 2. Arguments are direct adaptation from section 5 in [8].

On the basis of the boundary layer analysis, one can build an approximation of u^ε of boundary layer type. Namely, we introduce

$$u^\varepsilon_{app}(\omega,x) \;=\; u^0(x) \;+\; 6\,\phi\,\varepsilon\, v\left(\omega, \frac{x}{\varepsilon}\right) + 6\,\phi\,\varepsilon\, u^1(\omega,x) + 6\,\phi\,\varepsilon\, r^\varepsilon(\omega,x).$$

In this approximation, u^0 is the Poiseuille flow and $v(\omega,\cdot)$ is the boundary layer solution of (1.4). As v does not converge to zero at infinity, we add a large scale corrector u^1 satisfying:

$$\begin{cases} u^0\cdot\nabla u^1 + u^1\cdot\nabla u^0 - \Delta u^1 + \nabla p = 0,\ x\in\Omega, \\ \mathrm{div}\ u^1 \;=\; 0,\ x\in\Omega, \\ \displaystyle\int_0^1 u^1\cdot e_1 dx_2 = 0, \\ u^1|_{y_2=0} = 0, \quad u^1|_{y_2=1} = -(\alpha,0). \end{cases} \tag{2.5}$$

It is just a combination of a Couette and a Poiseuille flow: $u^1 = \alpha x_2\,(2-3x_2)\,e_1$. Still, this approximation does not vanish at the boundary, which explains the addition of another term $r^\varepsilon(\omega,x)$. It must satisfy

$$\begin{cases} r^\varepsilon(x_1,0) \;=\; 0, \\ r^\varepsilon(x_1,1) \;=\; v\left(\dfrac{x_1}{\varepsilon},\dfrac{1}{\varepsilon}\right) - (\alpha,0), \\ \mathrm{div}\ r^\varepsilon \;=\; 0,\ x\in\Omega. \end{cases} \tag{2.6}$$

This remainder can be taken small in the sense of

Proposition 5 *This problem possesses a (non unique) solution r^ε such that*

$$\sup_x \mathbb{E}\,\|r^\varepsilon\|^2_{H^2(B(x,1)\cap\Omega)} \;=\; O(\varepsilon|\ln\varepsilon|).$$

Proof: The proof of this result mimics the one of proposition 14 in [8]. The corrector r^ε can be chosen in the form

$$r^\varepsilon = \nabla^\perp \psi, \quad \psi = a(x_1)x_2^3 + b(x_1)x_2^2 + c(x_1)x_2 + d(x_1).$$

The streamfunction ψ is determined up to a constant and polynomial in x_2. Its coefficients have explicit dependence on $v - (\alpha, 0)$. Hence, the H^2 estimate on r^ε follows from the control of various terms involving $v - (\alpha, 0)$. For instance, one must bound the $L^2(P \times (-1, 1))$ norm of

$$\begin{aligned} \int_0^{x_1} v_2\left(\omega, \frac{t}{\varepsilon}, \frac{1}{\varepsilon}\right) dt \;&=\; \varepsilon \int_0^{x_1/\varepsilon} v_2\left(\omega, y_1, \frac{1}{\varepsilon}\right) dy_1 \\ &=\; \varepsilon \int_{\omega(x_1/\varepsilon)}^{1/\varepsilon} (v_1 - \alpha)\left(\omega, \frac{x_1}{\varepsilon}, y_2\right)\, dy_2 \\ &\quad -\; \varepsilon \int_{\omega(0)}^{1/\varepsilon} (v_1 - \alpha)(\omega, 0, y_2)\, dy_2 \;:= I^\varepsilon(\omega, x_1) \;-\; I^\varepsilon(\omega, 0) \end{aligned}$$

where the last equality comes from the Stokes formula. Using stationarity of π, we get

$$\mathbb{E} \left\| x_1 \mapsto \int_0^{x_1} v_2\left(\cdot, \frac{t}{\varepsilon}, \frac{1}{\varepsilon}\right) dt \right\|^2_{L^2(-1,1)} \;\le\; 4\mathbb{E}|I^\varepsilon(\cdot, 0)|^2.$$

Thanks to the refined estimate (2.4), we finally obtain

$$\begin{aligned} \mathbb{E}|I^\varepsilon(\cdot, 0)|^2 \;&\le\; C\left(\varepsilon^2\, \mathbb{E} \int_{\omega(0)}^1 |(v_1 - \alpha)(\cdot, 0, y_2)|^2 dy_2 \;+\; \varepsilon \int_1^{1/\varepsilon} \mathbb{E}\, |(v_1 - \alpha)(\cdot, 0, y_2)|^2 dy_2\right) \\ &\le\; C'\varepsilon^2 + C''\varepsilon \int_1^{1/\varepsilon} y_2^{-1}\, dy_2 = O(\varepsilon|\ln \varepsilon|) \end{aligned}$$

All other terms involve similar computations. These are straightforwardly adapted from the proof of proposition 14 in [8], using (2.4) instead of (2.3).

Once the approximate solution u^ε_{app} is built, one can obtain by energy estimates the following bounds, for ϕ small enough:

$$\begin{aligned} \|u^\varepsilon - u^\varepsilon_{app}\|_{L^2_{uloc}(P\times\Omega)} \;&=\; O\left(\varepsilon^{3/2}|\ln(\varepsilon)|^{1/2}\right), \\ \|u^\varepsilon_{app} - u^N\|_{L^2_{uloc}(P\times\Omega)} \;&=\; O\left(\varepsilon^{3/2}|\ln(\varepsilon)|^{1/2}\right), \end{aligned}$$

which of course imply theorem 2. As the proof is very similar to what was done in paper [8], we do not expand more and refer to it for all details.

3 A central limit theorem

Up to the end of the paper, we will assume (H1)-(H2), and focus on theorem 4. It is classical that (H2) implies ergodicity of π, so that the constant α does not depend on ω. We start again from an integral representation

$$\begin{aligned} \partial_y^\beta \left(v(\omega, 0, y_2) - (\alpha, 0)\right) \;&=\; \int_{\mathbb{R}} \partial_t^{\beta_1} \partial_{y_2}^{\beta_2} G(t, y_2)\, \left(v(\omega, -t, 0) - (\alpha, 0)\right)\, dt \\ &=\; -\int_{\mathbb{R}} \partial_t^{\beta_1+1} \partial_{y_2}^{\beta_2} G(t, y_2) \int_0^t \left(v(\omega, -s, 0) - (\alpha, 0))\right) ds\, dt, \end{aligned} \tag{3.1}$$

where the matrix kernel G is given by

$$G(y) \;=\; \frac{2y_2}{\pi(y_1^2+y_2^2)^2}\begin{pmatrix} y_1^2 & y_1y_2 \\ y_1y_2 & y_2^2 \end{pmatrix}$$

We introduce

$$V(\omega,t) \;:=\; \int_0^t \left(v(\omega,-s,0)-(\alpha,0)\right)\,ds.$$

A simple change of variable leads to

$$y_2^{|\beta|+1/2}\partial_y^\beta\left(v(\omega,0,y_2)-(\alpha,0)\right) \;=\; \int_{\mathbb{R}} \partial_t^{\beta_1+1}\partial_{y_2}^{\beta_2}G(u,1)\,y_2^{-1/2}\,V(\omega,y_2u)\,du. \tag{3.2}$$

Our first goal is to show that the l.h.s. converges in law to a gaussian distribution, for all β. We will focus on the case $|\beta|=0$, the other cases being handled in the exact same way. We state

Proposition 6 *The function V satisfies the following properties:*

i) $\mathbb{E}\,|V(\cdot,t)|^2 \;\le\; C|t|$

ii) *The random process $y_2^{-1/2}\,V(\omega,y_2\,u)$ converges weakly to a gaussian process $B(\omega,u)$ as y_2 goes to infinity.*

iii) *The covariance matrices also converge, that is for all indices i,j and for all s,t,*

$$\mathbb{E}\,y_2^{-1}\,V_i(\cdot,y_2\,s)\,V_j(\cdot,y_2\,t) \xrightarrow[y_2\to+\infty]{} \mathbb{E}B_i(\cdot,s)B_j(\cdot,t)$$

We remind that the process $X^n(\omega,t)$ with values in $\mathbb{R}^2$ *converges weakly* to $X(\omega,t)$ if, for all $T>0$ and all continuous bounded function $\mathcal{F}: C\left([-T,T],\mathbb{R}^2\right)\mapsto\mathbb{R}$,

$$\mathbb{E}\mathcal{F}(X^n) \xrightarrow[n\to+\infty]{} \mathbb{E}\mathcal{F}(X).$$

Theorem 4 is then a direct consequence of

Corollary 1 *The random process $y_2^{1/2}\left(v(\omega,0,y_2)-(\alpha,0)\right)$ converges in law to a gaussian vector with zero average. Moreover, for all i,j,*

$$\mathbb{E}\left(v_i(\cdot,0,y_2)-(\alpha,0)_i\right)\left(v_j(\cdot,0,y_2)-(\alpha,0)_j\right) \xrightarrow[y_2\to+\infty]{} \sigma_{ij},$$

where σ is the covariance matrix of this gaussian vector.

Proof of the corollary: To prove convergence in law to a gaussian vector $\mathcal{N}_\sigma$ of covariance matrix σ, we need to show that for any $F\in C_c^\infty(\mathbb{R})$,

$$\mathbb{E}\,F\left(\int_{\mathbb{R}}\partial_tG(t,1)\,y_2^{-1/2}\,V(\cdot,y_2t)\,dt\right) \xrightarrow[y_2\to+\infty]{} \mathbb{E}\,F\left(\mathcal{N}_\sigma\right).$$

Unsurprisingly, we take

$$\mathcal{N}_\sigma \;:=\; \int_{\mathbb{R}}\partial_tG(t,1)B(\omega,t)\,dt.$$

We decompose, for any $T > 0$,

$$
\begin{aligned}
&\mathbb{E}\, F\left(\int_{\mathbb{R}} \partial_t G(t,1)\, y_2^{-1/2}\, V(\cdot, y_2 t)\, dt\right) - \mathbb{E}\, F\left(\mathcal{N}_\sigma\right) \\
&= \mathbb{E}\, F\left(\int_{-T}^{T} \partial_t G(t,1)\, y_2^{-1/2}\, V(\cdot, y_2 t)\, dt\right) - \mathbb{E}\, F\left(\int_{-T}^{T} \partial_t G(t,1)\, B(\cdot, t)\, dt\right) \\
&+ \mathbb{E}\, F\left(\int_{\mathbb{R}} \partial_t G(t,1)\, y_2^{-1/2}\, V(\cdot, y_2 t)\, dt\right) - \mathbb{E}\, F\left(\int_{-T}^{T} \partial_t G(t,1)\, y_2^{-1/2}\, V(\cdot, y_2 t)\, dt\right) \\
&+ \mathbb{E}\, F\left(\int_{\mathbb{R}} \partial_t G(t,1)\, B(\cdot, t)\, dt\right) - \mathbb{E}\, F\left(\int_{-T}^{T} \partial_t G(t,1)\, B(\cdot, t)\, dt\right) \\
&:= J_1 + J_2 + J_3
\end{aligned}
$$

We now show that expressions J_1, J_2, converge to zero (J_3 is similar to J_2 and simpler). Let $\delta > 0$. We have

$$
\begin{aligned}
|J_2| \;&\le\; \max|F'|\, \mathbb{E} \int_{|t|>T} |\partial_t G(t,1)| \left| y_2^{-1/2}\, V(\omega, y_2 t)\right| \, dt \\
&\le\; \max|F'| \left(\int_{|t|>T} |\partial_t G(t,1)|\, dt\right)^{1/2} \left(\int_{|t|>T} |\partial_t G(t,1)|\; \mathbb{E}\left(y_2^{-1}\, |V(\omega, y_2 t)|^2\,\right) dt\right)^{1/2} \\
&\le\; C \left(\int_{|t|>T} |\partial_t G(t,1)|\, dt\right)^{1/2} \left(\int_{|t|>T} |\partial_t G(t,1)|\, t\, dt\right)^{1/2}
\end{aligned}
$$

where the last line comes from point ii) of proposition 6. Thus, for T large enough, independently of y_2, $|J_2| \le \delta/2$. Such T being fixed, for y_2 large enough, we get $|J_1| \le \delta/2$ by point i) of proposition 6. This yields convergence in law. The convergence of the covariance matrix

$$
\begin{aligned}
&\mathbb{E}\left(v_i(\cdot, 0, y_2) - (\alpha, 0)_i\right)\, \left(v_j(\cdot, 0, y_2) - (\alpha, 0)_j\right) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}\left(\partial_t G(s,1) y_2^{-1/2}\, V(\cdot, y_2\, s)\right)_i \left(\partial_t G(t,1) y_2^{-1/2}\, V(\cdot, y_2\, t)\right)_j \, ds\, dt
\end{aligned}
$$

follows from the dominated convergence theorem, using i) and iii) of proposition 6. We get

$$
\sigma_{ij} \;=\; \mathbb{E} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\partial_t G(s,1) B(\cdot, s)\right)_i \, \left(\partial_t G(t,1) B(\cdot, t)\right)_j \, ds\, dt.
$$

This concludes the proof of the corollary.

It remains to prove theorem 6. Note that point ii) is essentially a central limit theorem for the sequence of random variables

$$
X^n(\omega) = F \circ \tau_n(\omega), \quad F(\omega) = \int_0^1 \left(v(\omega, t, 0) - (\alpha(\omega), 0)\right)\, dt.
$$

The problem is that these random variables are not independent, due to "propagation of information at infinite speed" in the Stokes system. To establish a central limit theorem for such type of sequences is a classical question. The basic idea is that one can extend the central limit theorem to non independent sequences that feature a good decay of correlations as n goes

to infinity. We now illustrate this general principle on our problem, using assumption (H2). We follow the presentation of article [24], in which a similar question arises for a semilinear heat equation with random source. Let $\mathcal{C}^n$ the σ-algebra generated by the applications $y_1 \mapsto \omega(y_1)$, $|y_1| < n$.We state the following lemma:

Lemma 7 *Suppose that* $v^n := \mathbb{E}\left(v(\cdot,0,0) \mid \mathcal{C}^n\right)$ *satisfies*

$$\mathbb{E}\left|v^n - v(\cdot,0,0)\right|^2 \leq C\, n^{-\alpha}$$

for some $\alpha > 1$. *Then, proposition 6 holds.*

Proof of the lemma: We write the decomposition

$$v(\cdot,0,0) - (\alpha,0) = v^1 - (\alpha,0) + \sum_{j=1}^{+\infty}\left(v^{2^j} - v^{2^{j-1}}\right)$$

with the sum converging in $L^2(P)$. The corresponding sum for V is

$$V = \sum_{j=0}^{+\infty} V^j, \quad V^j(\omega,t) = \int_0^t \left(v^{2^j} - v^{2^{j-1}}\right) \circ \tau_s(\omega)\, ds,$$

where $v^{1/2} := (\alpha,0)$. Then, we have: $\|V(\cdot,t)\|_{L^2(P)} \leq \sum_{j=0}^{+\infty} \|V^j(\cdot,t)\|_{L^2(P)}$. By the assumption of independence at large distances, the correlations $\mathbb{E}\left(v^{2^j} \circ \tau_t(\omega)\, v^{2^j} \circ \tau_{t+s}(\omega)\right)$ and $\mathbb{E}\left(v^{2^j} \circ \tau_t(\omega)\, v^{2^{j-1}} \circ \tau_{t+s}(\omega)\right)$ vanish for $|s| \geq \kappa + 2^{j+1}$. We introduce

$$n := \left[|t|/(\kappa + 2^{j+1})\right].$$

If $n = 0$, we just write

$$\mathbb{E}\left|V^j(\cdot,t)\right|^2 \leq |t|^2 \mathbb{E}\left|v^{2^j} - v^{2^{j-1}}\right|^2.$$

If $n \geq 1$, we decompose

$$\begin{aligned}
\mathbb{E}\left|V^j(\cdot,t)\right|^2 &= \mathbb{E}\left|\sum_{k=0}^{n-1} \int_{kt/n}^{(k+1)t/n} \left(v^{2^j} \circ \tau_s - v^{2^{j-1}} \circ \tau_s\right)\, ds\right|^2 \\
&\leq \mathbb{E}\left|\int_0^{t/n} \sum_{k=0}^{n-1} \left(v^{2^j} \circ \tau_{s+kt/n} - v^{2^{j-1}} \circ \tau_{s+kt/n}\right)\, ds\right|^2 \\
&\leq 2\left(\kappa + 2^{j+1}\right) \int_0^{|t|/n} \sum_{k=0}^{n-1} \mathbb{E}\left|v^{2^j} - v^{2^{j-1}}\right|^2
\end{aligned}$$

Using the bound on the conditional expectations, we end up with

$$\|V^j(\cdot,t)\|^2_{L^2(P)} \leq C\,|t|\, \min(|t|, 2^j)\, 2^{-j\alpha}. \tag{3.3}$$

for some constant $C = C(\kappa)$. We thus get i). To prove ii), we just write the decomposition

$$v(\cdot,0,0) - (\alpha,0) = \sum_{j=0}^{+\infty}\left(v^{2^j} - v^{2^{j-1}}\right), \quad y_2^{-1/2}\, V(\omega, ty_2) = y_2^{-1/2} \sum_{j=0}^{+\infty} V^j(\omega, ty_2).$$

It is well-known that each finite sum satisfies a central limit theorem, that is

$$\forall k, \quad y_2^{-1/2} \sum_{j=0}^{k} V^j(\omega, ty_2) \xrightarrow[y_2 \to +\infty]{} B^k(\omega, t)$$

in the sense of weak convergence, to some gaussian process $B^k(\omega, t)$. Moreover, the covariance matrix also converges, that is

$$y_2^{-1} \mathbb{E} \sum_{j=0}^{k} V_l^j(\cdot, sy_2) \sum_{j=0}^{k} V_m^j(\cdot, ty_2) \xrightarrow[y_2 \to +\infty]{} \mathbb{E} B_l^k(\omega, s)\, B_m^k(\omega, t).$$

In short, it is due to the fact that the random variables

$$X^{n,j}(\omega) = F^j \circ \tau_n(\omega), \quad F^j(\omega) = \int_0^1 \left(v^{2^j} - v^{2^{j-1}} \right) \circ \tau_t(\omega)\, dt, \quad n \in \mathbb{Z},$$

have zero correlations at large distances: see [13, theorem (7.11) p424] for a similar result and detailed proof. Moreover, thanks to estimate (3.3), the remainder

$$R^k(\omega, t, y_2) \; = \; \sum_{j=k}^{+\infty} y_2^{-1/2}\, V^j(\omega, ty_2)$$

converges to zero as $k \to +\infty$, locally uniformly in t, uniformly in y_2. Hence, points (ii) and (iii) of proposition 6 hold, which ends the proof of the lemma.

We still have to estimate the variance of $v^n - v(\cdot, 0, 0)$. Following [24], we can turn this question into a question of domain of dependence for solutions of (1.4). Precisely, starting from the measure π on P, we define a new measure π^n on the product space

$$P^n \; = \; \{(\omega_1, \omega_2) \in P \times P, \quad \omega_1(t) = \omega_2(t), \; |t| \le n\}\,.$$

endowed with its cylindrical $\sigma-$field. Namely, π^n is defined in the following way:

1. $\pi^n(A \times A) := \pi(A)$, $\forall A \in \mathcal{C}^n$, which determines π^n over the sub $\sigma-$field $\mathcal{D}^n$ generated by the applications $t \mapsto (\omega_1(t), \omega_2(t))$, $|t| \le n$.
2. For all $k \ge 1$, for all $t^1, \dots, t^k$ with $|t^j| > n$, for all borelian subsets $B_1^1, \dots, B_1^k$, $B_2^1, \dots, B_2^k$ of $\mathbb{R}$, and

$$A_1 \; := \; \cap_{j=1}^k \{\omega_1, \; \omega_1(t_j) \in B_1^j\}, \quad A_2 \; := \; \cap_{j=1}^k \{\omega_2, \; \omega_2(t_j) \in B_2^j\}$$

   $\pi^n(A_1 \times A_2 \,|\, \mathcal{D}^n)(\omega_1, \omega_2) \; := \; \pi(A_1 \,|\, \mathcal{C}^n)(\omega_1)\, \pi(A_2 \,|\, \mathcal{C}^n)(\omega_2)$, which determines π^n conditionaly to $\mathcal{D}^n$.

It is then easy to derive the following identity, see [24]:

$$\mathbb{E}\, |v^n - v(\cdot, 0, 0)|^2 \; = \; \frac{1}{2} \int_{P^n} |v(\omega_1, 0, 0) - v(\omega_2, 0, 0)|^2 \, d\pi_n.$$

Thus, if $\Omega^{bl}(\omega_1)$ and $\Omega^{bl}(\omega_2)$ are two boundary layer domains with boundaries that coincide over $[-n, n]$, we need to estimate the difference of the corresponding boundary layer solutions $v(\omega_1, 0, 0)$ and $v(\omega_2, 0, 0)$. This is the purpose of the next section.

4 Decay of correlations

Throughout the rest of the paper, we will assume (H1). The main difficuly is to prove the following

Proposition 8 *Under assumption (H1), for all $0 < \tau < 1$, for almost every $\omega_1, \omega_2 \in P$,*

$$|v(\omega_1, 0, 0) - v(\omega_2, 0, 0)| \;\le\; \frac{C}{n^{2\tau - 1}}, \tag{4.1}$$

where C does not depend on ω_1, ω_2.

Together with the results of the preceding section, this proposition concludes the proof of theorem 4 (take $\tau > 3/4$), and therefore the proof of the main theorem 2. In fact, the sharper bound

$$|v(\omega_1, 0, 0) - v(\omega_2, 0, 0)| \;\le\; \frac{C}{n},$$

that is with $\tau = 1$ would still be true. We will discuss this briefly in the last section of the paper. For the sake of brevity, we only prove here the weaker form (4.1).

The main difficulty is that the boundary layer solutions $v(\omega_1, y)$ and $v(\omega_2, y)$ of (1.4) are not defined on the same domain, so that estimates on the difference are not directly available. If the Poisson equation rather than the Stokes system was considered, representation of the solution in terms of Brownian motion would allow to conclude quite easily. Again, this will be explained in the last section of the paper.

In the case of system (1.4), we are not aware of such representation, and the bound (4.1) will come from an accurate description of the (matrix) Green function of the Stokes operator above a humped boundary. We consider for all $\omega \in C^{2,\alpha}$, and for all $z \in \Omega^{bl}(\omega) = \{y_2 > \omega(y_1)\}$, the system:

$$\left\{ \begin{aligned} -\Delta G_\omega(z,\cdot) + \nabla P_\omega(z,\cdot) = \delta_z\, I_2 & \quad \text{in } \Omega^{bl}(\omega), \\ \text{div } G_\omega(z,\cdot) = 0 & \quad \text{in } \Omega^{bl}(\omega), \\ G_\omega(z,\cdot) = 0 & \quad \text{on } \partial\Omega^{bl}(\omega). \end{aligned} \right. \tag{4.2}$$

where δ_z is the Dirac mass at z, and I_2 is the 2×2 identity matrix. Let us remind how to build the matrix Green function (G_ω, P_ω). Up to a vertical translation of the domain, we can first assume that $z_2 > 0$. We then introduce the Green function (G_0, P_0) for the Stokes operator in the upper-half plane, see [14]. Extending $G_0(z,\cdot), P_0(z,\cdot)$ by 0 for $y_2 < 0$, the functions

$$H(z,\cdot) \;:=\; G_\omega(z,\cdot) - G_0(z,\cdot), \quad Q(z,\cdot) \;:=\; P_\omega(z,\cdot) - P_0(z,\cdot)$$

satisfy formally

$$\left\{ \begin{aligned} -\Delta H(z,\cdot) + \nabla Q(z,\cdot) = 0 & \quad \text{in } \Omega^{bl}(\omega), \\ \text{div } H(z,\cdot) = 0 & \quad \text{in } \Omega^{bl}(\omega), \\ H_\omega(z,\cdot) = 0 & \quad \text{on } \partial\Omega^{bl}(\omega), \\ [H(z,\cdot)] = 0, & \quad \left[\partial_2 H(z,\cdot) - Q(z,\cdot) \otimes e_2\right] = -\left[\partial_2 G_0(z,\cdot) - P_0(z,\cdot) \otimes e_2\right], \end{aligned} \right.$$

where $[\,\cdot\,]$ is the jump along $\{y_2 = 0\} \cap \Omega^{bl}(\omega)$. The jump on the derivative is explicit, as

$$\partial_2 G_0(z,(y_1,0^+)) - P_0(z,(y_1,0^+)) \otimes e_2 \;=\; \frac{2z_2}{\pi((z_1-y_1)^2 + z_2^2)^2} \begin{pmatrix} (z_1-y_1)^2 & (z_1-y_1)y_2 \\ (z_1-y_1)y_2 & y_2^2 \end{pmatrix}.$$

Standard variational formulation yields existence and uniqueness of a solution $H(z,\cdot)$ with $\nabla H(z,\cdot)$ in L^2. In turn, this provides a unique solution $G_\omega(z,\cdot)$ to (4.2). The corresponding pressure field $P_\omega(z,\cdot)$ is determined up to the addition of a constant matrix. Note that uniqueness yields the relation

$$G_{\tau_h(\omega)}(z,y) \;=\; G_\omega((z_1+h,z_2),(y_1+h,y_2)). \tag{4.3}$$

Our key estimate is provided by

Lemma 9 *For all $0<\tau<1$, for all $z,y\in\Omega^{bl}(\omega)$ satisfying $|z-y|\geq 1$, we have*

$$\sum_{|\beta|\leq 2} |\partial_y^\beta G_\omega(z,y)| \;+\; |\nabla_y P_\omega(z,y)| \;\leq\; C\,\frac{\delta(z)^\tau\,(1+\delta(y))^\tau}{|z-y|^{2\tau}}, \tag{4.4}$$

where $\delta(\cdot)$ denotes the distance to the boundary $\partial\Omega^{bl}(\omega)$, and C is a constant depending only on τ and on $\|\omega\|_{C^{2,\alpha}}$.

Note that by symmetry of G, we also have

$$\sum_{|\beta|\leq 2} |\partial_z^\beta G_\omega(z,y)| \;\leq\; C\,\frac{(1+\delta(z))^\tau\,(1+\delta(y))^\tau}{|z-y|^{2\tau}}. \tag{4.5}$$

Moreover, in the course of the proof of lemma 9, we will show that for all $y,z\in\Omega^{bl}(\omega)$,

$$|G_\omega(z,y)| \;\leq\; C\,\left(\big|\ln|z-y|\big|+1\right). \tag{4.6}$$

Let us show proposition 8, postponing the proof of lemma 9 to the next section. We first need to connect the solution $v(\omega,\cdot)$ of (1.4) to the Green function G_ω. For this purpose, we rather consider

$$w(\omega,y) \;=\; v(\omega,y) + y_2\,\mathbf{1}_{\{y_2<0\}}(y).$$

which satisfies (2.2). Note that v and w coincide at $y=0$. Formally, w should be equal to

$$\tilde w(\omega,z) \;=\; \int_{\{y_2=0\}} G_\omega(z,y)\,e_1\,dy. \tag{4.7}$$

Using estimates (4.4), (4.6), it is standard to show that $\tilde w$ is a solution of (2.2) in $H^1_{loc}(\overline{\Omega_{bl}})$. Using bound (4.5), one has even

$$\int_{\Omega^{bl}\cap\{|z_1|<1\}} |\nabla\tilde w|^2 \;\leq\; C \;<\; +\infty,$$

for all $\omega\in P_\alpha$.

Extending $\tilde w(\omega,\cdot)$ by 0 outside $\Omega^{bl}(\omega)$, one can show that $\omega\mapsto\tilde w(\omega,\cdot)$ is measurable from P_α to $H^1_{loc}(\mathbb{R}^2)$ (see the appendix for details). Moreover, thanks to (4.3), $\tilde w$ satisfies the stationarity relation $\tilde w(\tau_h(\omega),y)=\tilde w(\omega,(y_1+h,y_2))$. Finally, using that w and $\tilde w$ both satisfy (2.2), a simple energy estimate on the difference leads to

$$\mathbb{E}\int_{\Omega^{bl}\cap\{|z_1|<1\}} |\nabla(\tilde w - w)|^2 \;=\; 0$$

which shows that $w = \tilde{w}$ almost surely.

It remains to estimate the difference

$$v(\omega_1, 0, 0) - v(\omega_2, 0, 0) \;=\; \int_{\{y_2=0\}} \left(G_{\omega_1}((0,0), y) - G_{\omega_2}((0,0), y)\right)\, e_1,$$

for every ω_1, ω_2 in P_α which coincide over $[-n, n]$. This integral is bounded by

$$\begin{aligned} I_1 + I_2 \;:=\; & \int_{y_2=0, |y_1|>n} \left|G_{\omega_1} - G_{\omega_2}\right| ((0,0), y)\, dy \\ & + \int_{y_2=0, |y_1|\le n} \left|G_{\omega_1} - G_{\omega_2}\right| ((0,0), y)\, dy \end{aligned}$$

The use of (4.4) gives

$$I_1 \;\le\; C \int_{y_2=0, |y_1|>n} \frac{1}{|y_1|^{2\tau}}\, dy_1 \;\le\; \frac{C}{n^{2\tau-1}}.$$

where C, which depends *a priori* on $\|\omega_i\|_{C^{2,\alpha}}$, can be chosen uniformly over P_α, as all $C^{2,\alpha}$ norms are bounded by K_α. To bound the second term, we first assume that $\omega_2 > \omega_1$ for $|y_1| > n$, which is always possible up to introduce an intermediate third boundary. Hence, $\Omega^{bl}(\omega_2) \subset \Omega^{bl}(\omega_1)$. To lighten notations, we introduce

$$\Omega^{bl}_{1,2} \;:=\; \Omega^{bl}(\omega_1) \setminus \Omega^{bl}(\omega_2), \quad \Gamma_{1,2} \;:=\; \partial\Omega^{bl}(\omega_2) \setminus \partial\Omega^{bl}(\omega_1),$$

as well as

$$\tilde{P}(y) \;:=\; P_{\omega_2}\big((0,0), (y_1, \omega_2(y_1))\big), \quad y \in \Omega^{bl}_{1,2}$$

which defines a continuous extension of $P_{\omega_2}((0,0), \cdot)$ outside $\Omega^{bl}(\omega_2)$. Finally, we define the vector fields

$$\begin{aligned} U(y) \;&:=\; \left(G_{\omega_1} - G_{\omega_2}\right)((0,0), y), \quad & Q(y) \;&:=\; \left(P_{\omega_1} - P_{\omega_2}\right)((0,0), y), \quad y \in \Omega^{bl}(\omega_2), \\ U(y) \;&:=\; G_{\omega_1}((0,0), y), & Q(y) \;&:=\; P_{\omega_1}((0,0), y) - \tilde{P}(y), \quad y \in \Omega^{bl}_{1,2}. \end{aligned}$$

They satisfy

$$\begin{cases} -\Delta U + \nabla Q = 0, \quad y \in \Omega^{bl}(\omega_2), \\ -\Delta U + \nabla Q = -\nabla \tilde{P}, \quad y \in \Omega^{bl}_{1,2}, \\ \qquad \mathrm{div}\, U = 0, \quad y \in \Omega^{bl}(\omega_1), \\ \qquad\quad U = 0, \quad y \in \partial\Omega^{bl}(\omega_1), \\ \quad [U]\,|_{\Gamma_{1,2}} = 0, \quad [\partial_n U - Q \otimes n]\,|_{\Gamma_{1,2}} = -\partial_n G_{\omega_2}((0,0), y)|_{\Gamma_{1,2}}. \end{cases} \tag{4.8}$$

A direct energy estimate yields

$$\begin{aligned} & \int_{\Omega^{bl}(\omega_1)} |\nabla U|^2 \;\le\; \int_{\Gamma_{1,2}} |\partial_n G_{\omega_2}((0,0), y)|\, |U| \;+\; \int_{\Omega^{bl}_{1,2}} |\nabla \tilde{P}|\, |U| \\ & \le\; \Big(\int_{\Gamma_{1,2}} |\partial_n G_{\omega_2}((0,0), y)|^2\Big)^{1/2} \Big(\int_{\Gamma_{1,2}} |U|^2\Big)^{1/2} \;+\; \Big(\int_{\Omega^{bl}_{1,2}} |\nabla \tilde{P}|^2\Big)^{1/2} \Big(\int_{\Omega^{bl}_{1,2}} |U|^2\Big)^{1/2} \\ & \le\; C\left(\Big(\int_{\Gamma_{1,2}} |\partial_n G_{\omega_2}((0,0), y)|^2\Big)^{1/2} + \Big(\int_{\Omega^{bl}_{1,2}} |\nabla \tilde{P}|^2\Big)^{1/2}\right) \Big(\int_{\Omega^{bl}_{1,2}} |\partial_{y_2} U|^2\Big)^{1/2}. \end{aligned}$$

Note that all y in both $\Gamma_{1,2}$ and $\Omega^{bl}_{1,2}$ satisfy $|y_1| > n$. Using (4.4), we end up with

$$\int_{\Omega^{bl}(\omega_1)} |\nabla U|^2 \,\le\, C\, n^{1-4\tau}$$

Back to I_2, we obtain

$$|I_2| \,\le\, \sqrt{2n}\left(\int_{|y_1|\le n, y_2=0} |U|^2\right)^{1/2} \,\le\, C\sqrt{n}\left(\int_{\Omega^{bl}(\omega_1)} |\partial_{y_2} U|^2\right)^{1/2} \,\le\, \frac{C}{n^{2\tau-1}}.$$

This ends the proof of proposition (8).

5 Green function estimates

This section is devoted to the proof of lemma 9, that is sharp estimates on the Green function (G_ω, P_ω) for the Stokes operator above the humped boundary $y_2 = \omega(y_1)$, where ω belongs to $C^{2,\alpha}$. A fundamental remark is that the Green function satisfies the scaling

$$\forall \varepsilon > 0, \quad G_{\omega^\varepsilon}(\varepsilon z, \varepsilon y) \,=\, G_\omega(z,y), \quad \omega^\varepsilon(x_1) = \varepsilon\omega(x_1/\varepsilon). \tag{5.1}$$

We want estimates (4.4) to hold for $|z-y|$ large, that is for $\varepsilon \,:=\, |z-y|^{-1}$ small. By relation (5.1), to establish such estimates amounts to get local estimates for the Green function G_{ω^ε}. Thus, this is again a homogenization problem: more precisely, we must show that the oscillations of the boundary at frequency ε^{-1} do not affect too much the estimates on G_{ω^ε}, so that it behaves as the Green function for a half-plane. A very close problem has been considered in the papers [5, 6] by Avellaneda and Lin, namely the derivation of local estimates for elliptic systems div $(A(x/\varepsilon)\nabla\,\cdot)$, in which A is a positive definite matrix with periodic coefficients. Our reasoning follows these papers.

For all $x \in \mathbb{R}^2$, $r > 0$, we will denote $D(x,r)$ the disk of center x and radius r, and

$$D^\varepsilon(x,r) \,:=\, D(x,r) \cap \{x_2 > \omega^\varepsilon(x_1)\}, \quad \Gamma^\varepsilon(x,r) \,:=\, D(x,r)\cap\{x_2 = \omega^\varepsilon(x_1)\}.$$

An important property is that for all $0 < r < 1$,

$$|D^\varepsilon(x,r)| \,\ge\, \eta\, r^2, \tag{5.2}$$

for some $\eta > 0$ independent of ε. More precisely, η only involves the Lipschitz norm of ω^ε, wich is bounded uniformly in ε. This will be used implicitly throughout the sequel.

The core of the proof is to derive elliptic estimates uniform with respect to ε on the following Stokes problem:

$$\left\{\begin{aligned} -\Delta u + \nabla p \,=\, \text{div } f, &\quad x \in D^\varepsilon(x_0,1) \\ \text{div } u = 0, &\quad x \in D^\varepsilon(x_0,1), \\ u = 0, &\quad x \in \Gamma^\varepsilon(x_0,1), \end{aligned}\right. \tag{5.3}$$

where $x_0 \in \mathbb{R}^2$. More precisely, there are two steps in the proof:

1. We show a ε-uniform Hölder estimate on u: for all $f \in L^q$, $q > 2$ and for $\mu = 1 - 2/q$,

   $$\|u\|_{C^{0,\mu}(D^\varepsilon(x_0,1/2))} \,\le\, C\left(\|f\|_{L^q(D^\varepsilon(x_0,1))} + \|u\|_{L^2(D^\varepsilon(x_0,1))}\right). \tag{5.4}$$

   where C depends only on $\|\omega\|_{C^{1,\alpha}}$.

2. Thanks to this Hölder estimate, we prove (4.4).

The two next paragraphs correspond to these steps.

5.1 Hölder estimate

To obtain a Hölder regularity result, a classical approach is to use a characterization of Hölder spaces due to Campanato (see [16]): for Ω an open connected bounded set, $v \in C^{0,\mu}(\Omega)$ iff $v \in L^2(\Omega)$ and

$$\sup_{x\in\Omega, r>0} \frac{1}{r^{2+2\mu}} \int_{\Omega(x,r)} |v - \overline{v}_{x,r}|^2 < \infty, \quad \Omega(x,r) := \Omega \cap D(x,r), \;\; \overline{v}_{x,r} := \frac{1}{|\Omega(x,r)|} \int_{\Omega(x,r)} v.$$

One then tries to control such local integrals through energy estimates. This approach has been successful in the study of elliptic systems, see the work of Giaquinta and coauthors [16]. It extends to the Stokes type equations, *cf* article [17]. For us, it amounts to controlling

$$I^\varepsilon_{x,r} \;:=\; \frac{1}{r^{2+2\mu}} \int_{D^\varepsilon(x,r)} |u - \overline{u}_{x,r}|^2 < \infty, \quad \overline{u}_{x,r} := \frac{1}{|D^\varepsilon(x,r)|} \int_{D^\varepsilon(x,r)} u$$

where u is solution of (5.3). Note that, thanks to (5.2) (see [16]),

$$\|u\|_{C^{0,\mu}(D^\varepsilon(x_0,1/2))} \;\leq\; C_{x_0} \left(\|u\|_{L^2(D^\varepsilon(x_0,1/2))} \;+ \sup_{x\in D^\varepsilon(x_0,1/2), r>0} I^\varepsilon(x,r) \right)$$

with C_{x_0} independent of ε. In our case, the main problem is to keep track of the dependence of $I^\varepsilon_{x,r}$ on ε. It involves a discussion of the way ε relates to r. Broadly speaking, the idea is the following: if r is large compared to ε, then the oscillations have small enough amplitude to apply the regularity results of the flat case. On the contrary, when r gets as small or even smaller than ε, one can rescale everything by a factor ε, so that oscillations of the boundary have frequency $O(1)$, and are no longer annoying. Implementation of this idea is a bit technical, and follows closely the work of Avellaneda and Lin.

We first remind a few elements of regularity theory for Stokes type systems. Let Ω an open connected bounded set, with *Lipschitz boundary.* Then, for any $\varphi \in L^2(\Omega)$ satisfying $\int_\Omega \varphi = 0$, the problem

$$\mathrm{div}\; w = \varphi, \quad w|_{\partial\Omega} = 0$$

has one solution w satisfying $\|w\|_{H^1_0} \;\leq\; C\,\|\varphi\|_{L^2(\Omega)}$, where C *can be taken as an increasing function of* $|\Omega|$ *and of the Lipschitz constant* K *of the boundary*, see [14]. Thanks to this result, one has quite easily,

$$\|p - \int_\Omega p\|_{L^2(\Omega)} \;\leq\; C\,\|\Delta u + f\|\|_{H^{-1}(\Omega)} \tag{5.5}$$

where $(u,p) \in H^1(\Omega) \times L^2(\Omega)$ satisfies (in the distributional sense)

$$-\Delta u + \nabla p = f, \quad \mathrm{div}\; u = 0, \quad x \in \Omega. \tag{5.6}$$

Again, the constant C in (5.5) depends only on $|\Omega|$ and the Lipschitz constant of the boundary.

We now state the famous Cacciopoli inequality:

Lemma 10 *For all* $0 < r < 1$*, any solution* $u \in H^1(\Omega)$ *of* (5.3) *satisfies*

$$\|\nabla u\|_{L^2(D^\varepsilon(x,r))} \;\leq\; C\left(r^{-1}\,\|u\|_{L^2(D^\varepsilon(x,2r))} \;+\; r^\mu\,\|f\|_{L^q(D^\varepsilon(x,2r))}\right). \tag{5.7}$$

Sketch of proof: We remind the main elements of proof. Let η a smooth function with compact support in $D(x,2r)$, with $\eta = 1$ on $D(x,r)$. Note that $|\nabla \eta| \leq C r^{-1}$. Multiplying (5.3) by the test function $\eta^2 u$, and integrating by parts, one has easily

$$\int_{D^\varepsilon(x,r)} |\nabla u|^2 \;\leq\; \int_{D^\varepsilon(x,2r)} \eta^2 |\nabla u|^2 \;\leq\; C\, r^{-2} \int_{D^\varepsilon(x,2r)} |u|^2 \;+\; C \int_{D^\varepsilon(x,2r)} |f|^2 \\ \;+\; \|p - \overline{p}_{x,2r}\|_{L^2(D^\varepsilon(x,2r))} \, \|\text{div } (\eta u)\|_{L^2(D^\varepsilon(x,2r))}.$$

Using (5.5), we get

$$\|p - \overline{p}_{x,2r}\|_{L^2(D^\varepsilon(x,2r))} \leq C\, \|\Delta u + \text{div } f\|_{H^{-1}(D^\varepsilon(x,2r))} = C \|v\|_{H^1(D^\varepsilon(x,2r))}$$

where $v \in H^1_0(D^\varepsilon(x,2r))$ is the solution of

$$-\Delta v + \nabla p = \Delta u + \text{div } f, \quad \text{div } v = 0, \quad v|_{\partial D^\varepsilon(x,2r)} = 0.$$

Note that the previous bound is uniform in ε, as it only involves the Lipschitz constant of ω^ε which is uniformly bounded. A simple energy estimate on v gives

$$\|\nabla v\|_{L^2(D^\varepsilon(x,2r))} \;\leq\; C \left(\|\nabla u\|_{L^2(D^\varepsilon(x,2r))} \;+\; \|f\|_{L^2(D^\varepsilon(x,2r))} \right)$$

As div $(\eta u) = \nabla \eta \cdot u$, and using Hölder inequality on f, we end up with

$$\int_{D^\varepsilon(x,r)} |\nabla u|^2 \;\leq\; \int_{D^\varepsilon(x,2r)} \eta^2 |\nabla u|^2 \;\leq\; C\, r^{-2} \int_{D^\varepsilon(x,2r)} |u|^2 \;+\; C_\delta\, r^{2\mu}\, \|f\|^2_{L^q(D^\varepsilon(x,2r))} \\ \;+\; \delta \|\nabla u\|^2_{L^2(D^\varepsilon(x,2r))},$$

where $\delta > 0$ is arbitrary small. We conclude as in [17, Theorem 1.1, page 180].

Inequality of type (5.7) has been used by Giaquinta and Modica in the study of elliptic regularity. In the context of Stokes type system, they obtain a local estimate, see [17]:

Theorem 11 *Let Ω of class C^1, and $(u,p,f) \in H^1(\Omega) \times L^2(\Omega) \times L^q(\Omega)$, $q > 2$, satisfying*

$$-\Delta u + \nabla p = div\ f,\ div\ u = 0, \quad x \in \Omega(x_0, 1), \quad u|_{\partial\Omega \cap D(x_0,1)} = 0.$$

Then, $u \in C^{0,\mu}(\Omega(x_0, 1/2))$ for $\mu = 1 - 2/q$, and

$$\|u\|_{C^{0,\mu}(\Omega(x_0,1/2))} \;\leq\; C \left(\|u\|_{L^2(\Omega(x_0,1))} \;+\; \|f\|_{L^q(\Omega(x_0,1))} \right). \tag{5.8}$$

Unfortunately, we cannot use this theorem assuch. Indeed, the constant C in the last regularity estimate involves the modulus of continuity of $\nabla\gamma$, where $x_2 = \gamma(x_1)$ describes the boundary. In our case $\gamma = \omega^\varepsilon$, such modulus of continuity is not uniformly bounded in ε. We must proceed in several steps to control the local integrals $I^\varepsilon_{x,r}$. Note that theorem 11 implies estimate (5.4) when $D^\varepsilon(x_0, 1)$ is far from the boundary. Thus, we can restrict ourselves to a case in which x_0 is close to the oscillating boundary, for instance belongs to the axis $x_2 = 0$.

Lemma 12 *For all θ small enough, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, and for all solutions of* (5.3) *satisfying* $\|u^\varepsilon\|_{L^2(D^\varepsilon(x_0,1/4))} \leq 1,\ \|f\|_{L^q(D^\varepsilon(x_0,1/4))} \leq \varepsilon_0$, *one has*

$$\|u^\varepsilon\|_{L^2(D^\varepsilon(x_0,\theta))} \;\leq\; \theta^{\mu+1}.$$

Proof of the lemma: Suppose that the result does not hold. Then one can find θ arbitrary small, and sequences u^{ε_j}, f^j satisfying

$$\|u^{\varepsilon_j}\|_{L^2(D^{\varepsilon_j}(x_0,1/4))} \le 1, \quad \|f^j\|_{L^2(D^{\varepsilon_j}(x_0,1/4))} \xrightarrow[j\to+\infty]{} 0, \quad \|u^{\varepsilon_j}\|_{L^2(D^{\varepsilon_j}(x_0,\theta))} > \theta^{\mu+1}.$$

One can extend all the u^ε_j, f^j by 0 outside $D^{\varepsilon_j}(x_0,1/4)$ so that all these functions are defined on the fixed domain $D(x_0,1/4)$. From the L^2 bound on u^{ε_j}, up to extract a subsequence, we get

$$u^{\varepsilon_j} \text{ converges weakly to some } u \text{ in } L^2(D(x_0,1/4))$$

and by Cacciopoli inequality (5.7),

$$u^{\varepsilon_j} \text{ converges weakly to } u \text{ in } H^1(D(x_0,1/8)), \text{ and strongly in } L^2(D(x_0,1/8)).$$

One can then take the limit in (5.3), which yields

$$-\Delta u + \nabla p = 0, \quad \text{div } u = 0, \quad \text{in } D(x_0,1/8)\cap\{x_2>0\}, \quad u|_{D(x_0,1/8)\cap\{x_2=0\}} = 0.$$

As the upper half plane is a regular domain, we can apply theorem 11, so that for all $\tilde\mu > \mu$, for all θ,

$$\begin{aligned}\|u\|_{L^2(D(x_0,\theta)\cap\{x_2>0\})} &\le 2\pi\, \|u\|_{C^{0,\tilde\mu}(D(x_0,\theta)\cap\{x_2>0\})}\, \theta^{\tilde\mu+1} \\ &\le C\, \|u\|_{L^2(D(x_0,1/8)\cap\{x_2>0\})}\, \theta^{\tilde\mu+1} \le C\, \theta^{\tilde\mu+1}.\end{aligned}$$

For θ small enough, it contradicts the lower bound on $\|u^{\varepsilon_j}\|_{L^2(D^{\varepsilon_j}(x_0,\theta))}$.

We fix θ, ε_0 as in lemma 12. We then state

Lemma 13 *For all ε, k satisfying $\varepsilon/\theta^k \le \varepsilon_0$ ($k \ge 1$),*

$$\int_{D^\varepsilon(0,\theta^k)} |u^\varepsilon|^2 \le \theta^{2k\mu+2} \left(\|u^\varepsilon\|_{L^2(D^\varepsilon(0,1/4))} + \frac{1}{\varepsilon_0} \|f\|_{L^2(D^\varepsilon(0,1/4))} \right)^2 \tag{5.9}$$

Proof of the lemma: The lemma is deduced from an induction argument on k. For $k=1$, the bound (5.9) is given by lemma 12. Assume now that this bound holds for $k \ge 1$. Up to a horizontal translation, we can assume that $x_0 = 0$. Then, we set

$$M := \|u^\varepsilon\|_{L^2(D^\varepsilon(0,1/4))} + \frac{1}{\varepsilon_0} \|f\|_{L^2(D^\varepsilon(0,1/4))},$$

and introduce the rescaled functions

$$v := \theta^{-k\mu} M^{-1} u(\theta^k x), \quad g = \theta^{k-k\mu} M^{-1} f(\theta^k x).$$

They satisfy

$$-\Delta v + \nabla q = \text{div } g, \quad \text{div } v = 0,\ x \in D^\varepsilon(0,\theta^{-k}/4), \quad v|_{\Gamma^\varepsilon(0,\theta^{-k}/4)}.$$

Moreover, one has $\|f\|_{L^q(D^\varepsilon(0,1/4))} \le \varepsilon_0$, and by the induction assumption

$$\|v\|_{L^2(D^\varepsilon(0,1/4))} \le 1.$$

Applying lemma 12 to v and g yields the result.

We can now finish the proof of estimate (5.4). Let $x \in D^\varepsilon(x_0,1/2)$. We need to bound $I^\varepsilon(x,r)$, for $r>0$. There are two cases to handle:

- *The distance between x and the boundary $\{x_2 = \omega^\varepsilon(x_1)\}$ satisfies $\delta^\varepsilon(x) \ge \frac{\varepsilon}{\varepsilon_0}$.*

Up to take a smaller ε_0, we can suppose that $\frac{\varepsilon}{\varepsilon_0} > \varepsilon$, which implies that there exists x_0' on the axis $\{x_2 = 0\}$ with $|x - x_0'| \le \delta^\varepsilon(x)$. By lemma 13, for all $\varepsilon/\varepsilon_0 \le r \le 1/12$,

$$\int_{D^\varepsilon(x_0',3r)} |u^\varepsilon|^2 \;\le\; C\, r^{2\mu+2} \left(\|u^\varepsilon\|_{L^2(D^\varepsilon(x_0',1/4))} \;+\; \|f\|_{L^q(D^\varepsilon(x_0',1/4))} \right)^2. \tag{5.10}$$

If $r > \delta^\varepsilon(x)/2$, $D^\varepsilon(x,r) \subset D^\varepsilon(x_0', 3r)$, and the previous line implies

$$\int_{D^\varepsilon(x,r)} |u^\varepsilon|^2 \;\le\; C\, r^{2\mu+2} \left(\|u^\varepsilon\|_{L^2(D^\varepsilon(x_0',1/4))} \;+\; \|f\|_{L^q(D^\varepsilon(x_0',1/4))} \right)^2.$$

On the contrary, if $r \le \delta^\varepsilon(x)/2$, then $D^\varepsilon(x,r) \;=\; D(x,r)$ (it does not intersect the boundary). A simple rescaling of (5.8) yields

$$\|u\|_{C^{0,\mu}(D(x,\delta^\varepsilon(x)/2))} \;\le\; C \left(\delta^\varepsilon(x)^{-1-\mu} \|u^\varepsilon\|_{L^2(D(x,\delta^\varepsilon(x)))} \;+\; \|f\|_{L^q(D(x,\delta^\varepsilon(x)))} \right).$$

Thus,

$$\int_{D^\varepsilon(x,r)} |u^\varepsilon|^2 \;\le\; C\, r^{2\mu+2} \left(\delta^\varepsilon(x)^{-1-\mu} \|u^\varepsilon\|_{L^2(D(x,\delta^\varepsilon(x)))} \;+\; \|f\|_{L^q(D(x,\delta^\varepsilon(x)))} \right)^2.$$

Now, by lemma 13, as $\delta^\varepsilon(x) \ge \varepsilon/\varepsilon_0$,

$$\begin{aligned} \|u^\varepsilon\|_{L^2(D(x,\delta^\varepsilon(x)))} \;&\le\; \|u^\varepsilon\|_{L^2(D(x_0',2\delta^\varepsilon(x)))} \\ &\le\; C\, \delta^\varepsilon(x)^{\mu+1} \left(\|u^\varepsilon\|_{L^2(D^\varepsilon(x_0',1/4))} \;+\; \|f\|_{L^q(D^\varepsilon(x_0',1/4))} \right). \end{aligned}$$

Using the two last inequalities, we end up again with

$$\int_{D^\varepsilon(x,r)} |u^\varepsilon|^2 \;\le\; C\, r^{2\mu+2} \left(\|u^\varepsilon\|_{L^2(D^\varepsilon(x_0',1/4))} \;+\; \|f\|_{L^q(D^\varepsilon(x_0',1/4))} \right)^2,$$

which in turn clearly implies

$$\int_{D^\varepsilon(x,r)} |u^\varepsilon - \overline{u^\varepsilon}_{x,r}|^2 \;\le\; C\, r^{2\mu+2} \left(\|u^\varepsilon\|_{L^2(D^\varepsilon(x_0',1/4))} \;+\; \|f\|_{L^q(D^\varepsilon(x_0',1/4))} \right)^2,$$

As $D^\varepsilon(x_0', 1/4)) \;\subset\; D^\varepsilon(x_0, 1)$, this gives the required estimate.

- *The distance between x and the boundary $\{x_2 = \omega^\varepsilon(x_1)\}$ satisfies $\delta^\varepsilon(x) < \frac{\varepsilon}{\varepsilon_0}$.*

This time, there exists x_0' on the axis $\{x_2 = 0\}$ such that $|x - x_0'| \le \delta^\varepsilon(x) + \varepsilon \;\le\; 2\varepsilon/\varepsilon_0$. Again, for all $\varepsilon/\varepsilon_0 \le r \le 1/12$, $D(x,r) \subset D(x_0', 3r)$ and (5.10) implies

$$\int_{D^\varepsilon(x,r)} |u^\varepsilon|^2 \;\le\; C\, r^{2\mu+2} \left(\|u^\varepsilon\|_{L^2(D^\varepsilon(x_0',1/4))} \;+\; \|f\|_{L^q(D^\varepsilon(x_0',1/4))} \right)^2.$$

It remains to handle the case $r < \varepsilon/\varepsilon_0$. Up to a horizontal translation, we can assume that $x_0' = 0$. We introduce the rescaled functions

$$v \;=\; \left(\frac{\varepsilon}{\varepsilon_0}\right)^{-\mu} u^\varepsilon\left(\frac{\varepsilon}{\varepsilon_0} x\right), \quad g \;=\; \left(\frac{\varepsilon}{\varepsilon_0}\right)^{1-\mu} f\left(\frac{\varepsilon}{\varepsilon_0} x\right).$$

They satisfy in particular

$$-\Delta v + \nabla q = \text{div } g, \quad \text{div } v = 0, \; x \in D^{\varepsilon_0}(0,1), \quad v|_{\Gamma^{\varepsilon_0}(0,1)} = 0. \tag{5.11}$$

It is a Stokes type system set in a domain independent of the small parameter ε. Hence, we can apply theorem 11, which yields: for all $r < 1$,

$$\int_{D^{\varepsilon_0}(x,r)} |v - \overline{v}_{x,r}|^2 \le C \, \|v\|_{C^{0,\mu}(D^{\varepsilon_0}(x,r))} \, r^{2\mu+2} \le C \, r^{2\mu+2} \, \left(\|v\|_{L^2(D^{\varepsilon_0}(0,2))} \, + \, \|g\|_{L^2(D^{\varepsilon_0}(0,2))} \right)^2 .$$

Back to the original unknowns u^ε, f, we obtain the control of $I^\varepsilon_{x,r}$ for $r \le \varepsilon/\varepsilon_0$. This ends the proof.

5.2 Bounds on the Green function

From the above Hölder estimate, we can deduce the estimate (4.4). The arguments are again adapted from article [5, pages 819,829-831]. For the sake of completeness, we describe the ideas at play. The first step is to establish the following bound: for all x, x' in $\{x_2 > \omega^\varepsilon(x_1)\}$,

$$|G_{\omega^\varepsilon}(x,x')| \; \le \; C \, \left(\big| \ln |x - x'| \big| \, + \, 1 \right). \tag{5.12}$$

where C only involves $\|\omega\|_{C^{1,\alpha}}$. Note that it implies (4.6). To lighten the notations, we drop the suffix ω, denoting G^ε, G instead of $G_{\omega^\varepsilon}, G_\omega$. Let us introduce the Green function $\tilde{G}^\varepsilon(x,t,x',t')$ for the Stokes operator over $\{x_2 > \omega^\varepsilon(x_1)\} \times \mathbb{T}$. Namely, it satisfies for all $(x,t) \in \{x_2 > \omega^\varepsilon(x_1)\} \times \mathbb{T}$

$$\left\{ \begin{aligned} -\Delta \tilde{G}^\varepsilon(x,t,\cdot) + \nabla \tilde{P}^\varepsilon(x,t,\cdot) \; &= \; \delta_{x,t} I_3 && \text{in } \{x_2 > \omega^\varepsilon(x_1)\} \times \mathbb{T}, \\ \text{div } \tilde{G}^\varepsilon(x,t,\cdot) = 0 & && \text{in } \{x_2 > \omega^\varepsilon(x_1)\} \times \mathbb{T}, \\ \tilde{G}^\varepsilon(x,t,\cdot) = 0 & && \text{on } \{x_2 = \omega^\varepsilon(x_1)\} \times \mathbb{T}. \end{aligned} \right. \tag{5.13}$$

One has easily that

$$G^\varepsilon(x,x') \; = \; \int_{\mathbb{T}} \left(\tilde{G}^\varepsilon_1(x,0,x',t'), \tilde{G}^\varepsilon_2(x,0,x',t') \right) \, dt'.$$

The point is to show that

$$|\tilde{G}^\varepsilon(x,t,x',t')| \; \le \; C \, \frac{1}{|x - x'| + |t - t'|}.$$

The estimate (5.12) is then obtained by integration with respect to t', at $t = 0$. Such bound on $\tilde{G}^\varepsilon$ will be deduced from a repeated use of the Hölder estimate (5.4). Note that this estimate extends without difficulty to similar Stokes problems in dimension $n \ge 2$, with $q > n$ and $\mu = 1 - n/q$. In particular, it holds when the domain is $\{x_2 > \omega^\varepsilon(x_1)\} \times \mathbb{T}$.

Namely, let $\tilde{x} := (x,t)$ and $\tilde{x}' := (x',t')$. Let $r := |\tilde{x} - \tilde{x}'|$, and $f \in C^\infty_c(D^\varepsilon(\tilde{x}', r/3))$. We consider the quantity

$$u^\varepsilon(\tilde{x}) \; = \; \int_{D^\varepsilon(\tilde{x}',r/3)} \tilde{G}^\varepsilon(\tilde{x},\tilde{z}) \, f(\tilde{z}) \, d\tilde{z}$$

The field u^ε satisfies a Stokes equation with source term f over $\{x_2 > \omega^\varepsilon(x_1)\} \times \mathbb{T}$, with a Dirichlet boundary condition. We therefore apply the estimate (5.4) to u^ε. Properly rescaled, it yields

$$|u^\varepsilon(\tilde{x})| \le C \left(r^{-3} \int_{D^\varepsilon(\tilde{x}, r/3)} |u^\varepsilon|^2 \right)^{1/2},$$

where we used the fact that f vanishes over $D^\varepsilon(\tilde{x}, r/3)$. Thus, we get

$$\begin{aligned}
&\left| \int_{D^\varepsilon(\tilde{x}', r/3)} \tilde{G}^\varepsilon(\tilde{x}, \tilde{z})\, f(\tilde{z})\, d\tilde{z} \right| \le C \left(r^{-3} \int_{D^\varepsilon(\tilde{x}, r/3)} |u^\varepsilon|^2 \right)^{1/2} \\
&\le C' \left(r^{-3} \int_{D^\varepsilon(\tilde{x}, r/3)} |u^\varepsilon|^6 \right)^{1/6} \le C'\, r^{-1/2} \left(\int_{D^\varepsilon(\tilde{x}, r/3)} |u^\varepsilon|^6 \right)^{1/6} \\
&\le C''\, r^{-1/2} \left(\int_{\{x_2 > \omega^\varepsilon(x_1)\} \times \mathbb{T}} |\nabla u^\varepsilon|^2 \right)^{1/2} \le C''\, r^{1/2} \|f\|_{L^2}^{1/2}
\end{aligned}$$

where the last two inequalities come respectively from the Sobolev imbedding theorem (note that u^ε is zero at the boundary so that the imbedding does not involve lower order terms), and from the standard energy estimate on the Stokes system. By duality, we infer that

$$\left(r^{-3} \int_{D^\varepsilon(\tilde{x}', r/3)} |\tilde{G}^\varepsilon(\tilde{x}, \tilde{z})|^2\, d\tilde{z} \right)^{1/2} \le C\, r^{-1}.$$

Using that $\tilde{G}^\varepsilon(\tilde{x}, \cdot)$ satisfies a homogenenous Stokes system over $D(\tilde{x}', r/3)$, one more application of (5.4) leads to

$$\tilde{G}^\varepsilon(\tilde{x}, \tilde{y}) \le C \left(r^{-3} \int_{D^\varepsilon(\tilde{x}', r/3)} |\tilde{G}^\varepsilon(\tilde{x}, \tilde{z})|^2\, d\tilde{z} \right)^{1/2} \le C\, r^{-1}.$$

Inequality (5.12) at hand, we can derive the final estimate on G. Let $x, x' \in \{x_2 > \omega^\varepsilon(x_1)\}$. Set this time $r := |x - x'|$. For all $\bar{x}$ such that $|\bar{x} - x| < 2r$, (5.12) implies

$$\left| G^\varepsilon(\bar{x}, x') \right| \le C \left(\left| \ln |\bar{x} - x'| \right| + 1 \right) \le C' \left(\left| \ln |x - x'| \right| + 1 \right).$$

Applying (5.4) to the function $G^\varepsilon(\cdot, x')$, we get for any $\tau \in (0,1)$

$$\begin{aligned}
\left| G^\varepsilon(x, x') \right| &\le \delta^\varepsilon(x)^\tau\, \|G^\varepsilon(\cdot, x')\|_{C^{0,\tau}(D(x, r/3))} \\
&\le C_\tau\, \delta^\varepsilon(x)^\tau\, r^{-1-\tau}\, \|G^\varepsilon(\cdot, x')\|_{L^2(D(x, 2r/3))}
\end{aligned}$$

that leads to

$$\left| G^\varepsilon(x, x') \right| \le C_\tau \left(\left| \ln |x - x'| \right| + 1 \right) \frac{\delta^\varepsilon(x)^\tau}{|x - x'|^\tau}.$$

Now reversing the roles of x and x', we obtain

$$|G^\varepsilon(x, x')| \le C_\tau \left(\left| \ln |x - x'| \right| + 1 \right) \frac{\delta^\varepsilon(x)^\tau\, \delta^\varepsilon(x')^\tau}{|x - x'|^{2\tau}}.$$

Using the scaling relation (5.1), we get for all $y, z \in \{y_2 > \omega(y_1)\}$, for $\varepsilon := |y-z|^{-1}$,

$$G(z,y) = G^\varepsilon(\varepsilon z, \varepsilon y) \le C_\tau \left(\left|\ln(\varepsilon|z-y|)\right| + 1\right) \frac{\delta^\varepsilon(\varepsilon z)^\tau \, \delta^\varepsilon(\varepsilon y)^\tau}{|\varepsilon(y-z)|^{2\tau}} = C_\tau \frac{\delta(z)^\tau \, \delta(y)^\tau}{|y-z|^{2\tau}}.$$

Using classical local regularity results for the Stokes equation in a $C^{2,\alpha}$ domain (see [17, Theorem 1.3, page 198], which extends theorem 11): for $|z-y| \ge 1$,

$$\begin{aligned} \sum_{|\beta|\le 2} |\partial_y^\beta G(z,y)| \; + \; |\nabla_y P(z,y)| \; &\le \; C \, \|G(z,\cdot)\|_{L^2(D(y,1/2))} \\ &\le \; C \, \frac{\delta(z)^\tau \, (1+\delta(y))^\tau}{|z-y|^{2\tau}} \end{aligned}$$

that is exactly estimate (4.4).

6 Comments

6.1 Well-posedness of the boundary layer system

As mentioned several times in this paper, the well-posedness of system (1.4) is not known without a structural assumption on ω, like periodicity or stationarity. We stress however that thanks to our estimates on the Green function G_ω, the representation formula (4.7) defines a solution of (1.4) for any $C^{2,\alpha}$ boundary, *cf* the fourth section. Hence, the open issue is rather to find the appropriate functional space for uniqueness.

Such difficulty does not arise when the Stokes operator is replaced by the Laplacian, or more generally by a scalar elliptic operator. Hence, one can show well-posedness in L^∞ of

$$-\Delta v = 0 \text{ in } \Omega^{bl}, \quad v(y) = \omega(y_1) \text{ on } \partial\Omega^{bl} \tag{6.1}$$

if the function ω is bounded and Lipschitz. For the existence part, one may consider, for all $n \ge 1$, the solution v^n of

$$-\Delta v^n = 0 \text{ in } \Omega^{bl} \cap D(0,n), \quad v^n(y) = \omega(y_1) \text{ on } \partial\left(\Omega^{bl} \cap D(0,n)\right).$$

By the maximum principle, $\|v^n\|_{L^\infty} \le \|\omega\|_{L^\infty}$, so that up to a subsequence it converges to some v in L^∞ weak*. Straightforwardly, v satisfies (6.1).

For the uniqueness part, let $v \in L^\infty$ satisfying

$$-\Delta v = 0 \text{ in } \Omega^{bl}, \quad v(y) = 0 \text{ on } \partial\Omega^{bl}$$

Let us show that $v = 0$. As we do not know the behavior of v at infinity, it does not follow directly from the maximum principle. In the case of a Lipschitz boundary ω, we can conclude to the uniqueness in the following way. By the change of variables $y := (y_1, \, y_2 - \omega(y_1))$, the previous equation becomes

$$\text{div } (A(y)\nabla v) = 0, \; y_2 > 0, \quad v|_{y_2=0} = 0,$$

for some elliptic matrix $A = (a_{ij})$ with bounded coefficients. We extend A and v to $\{y_2 < 0\}$ by the formulas, $v(y_1, y_2) := -v(y_1, -y_2)$,

$$\begin{aligned} &a_{in}(y_1,y_2) := -a_{in}(y_1,-y_2), \quad a_{nj}(y_1,y_2) := -a_{nj}(y_1,-y_2), \quad i,j \ne n, \\ &a_{ij}(y_1,y_2) := a_{ij}(y_1,-y_2) \quad \text{otherwise.} \end{aligned}$$

In this way, we get div $(A(y)\nabla v) = 0$ on all $\mathbb{R}^2$. Harnack's inequality for elliptic equations (see [18]) leads to

$$\sup_{|y|<R} (M+v) \;\le\; C \inf_{|y|<R} (M+v)$$

for any $R > 0$ and M such that $M + u \ge 0$. With $M = max(0, -\inf v)$ and R going to infinity, we obtain that $v = 0$.

6.2 Decay of correlations

A key element in the paper is the estimate (4.4) on the Green function for the Stokes operator above an oscillating boundary. This estimate relies itself on the Hölder regularity result (5.4). In fact, with some more calculations in the same spirit, one could show a refined bound: for a $C^{1,\alpha}$ boundary ω and a source term $f \in C^{0,\mu}(D^\varepsilon(0,1))$ $(\alpha, \mu > 0)$, the solution u^ε of (5.3)satisfies

$$\|\nabla u^\varepsilon\|_{L^\infty(D^\varepsilon(0,1/2))} \;\le\; C\left(\|u\|_{L^2(D^\varepsilon(0,1))} \;+\; \|f\|_{C^{0,\mu}(D^\varepsilon(0,1))}\right),$$

with C independent of ε. This yields an optimal bound on the Green function, that is

$$\sum_{|\beta|\le 2} |\partial_y^\beta G_\omega(z,y)| \;+\; |\nabla_y P_\omega(z,y)| \;\le\; C\,\frac{\delta(z)\,(1+\delta(y))}{|z-y|^2}. \tag{6.2}$$

Estimate (4.1) can in turn be improved as

$$|v(\omega_1,0,0) - v(\omega_2,0,0)| \;\le\; C\,n^{-1}.$$

Such bound is far easier to prove when (1.4) is replaced by the scalar system (6.1). In this case, one may use a representation in terms of the standard two-dimensional Brownian motion $B(m,t) = (B_1(m,t), B_2(m,t))$. If we denote $(M, \mathcal{M}, \mu)$ the probability space on which this Brownian motion is defined,

$$v(\omega,0,0) \;=\; \int_M (-\omega,0)\left(B_1(m,\tau(m))\right) d\mu$$

where τ is the exit time from $\Omega^{bl}(\omega)$ (see [25]). We now want to bound

$$v(\omega_1,0,0) - v(\omega_2,0,0) = \int_M \Big(\omega_1\big(B_1(m,\tau_1(m))\big) \;-\omega_2\big(B_1(m,\tau_2(m))\big)\Big)d\mu$$

where τ_i is the exit time from $\Omega^{bl}(\omega_i)$. We remind that $\omega_1 = \omega_2$ over $[-n,n]$, so the exit times are the same for brownian particles leaving in the region $y_1 \in [-n,n]$. Hence,

$$\begin{aligned}|v(\omega_1,0,0) - v(\omega_2,0,0)| \;&\le\; 2\max_{i=1,2}\|\omega_i\|_{L^\infty}\,\mu\left(|B_1(\cdot,\tau_i)| \ge n\right)\\ &\le\; 2\Big(\mu\left(T_{-n} \le T_{-1}\right) + \mu\left(T_n \le T_{-1}\right)\Big)\end{aligned}$$

where we denote by $T_{\pm n}$ the first time for which $B_1 = \pm n/2$, and T_{-1} the first time for which $B_2 = -1$. It is well-known that the distributions of these hitting times are

$$dT_{\pm n}(\mu) \;=\; \frac{n}{4\sqrt{2\pi t^3}}\exp\left(\frac{-n^2}{8t}\right)\mathbf{1}_{t>0}\,dt, \quad dT_{-1}(\mu) \;=\; \frac{1}{\sqrt{2\pi t^3}}\exp\left(\frac{-1}{2t}\right)\mathbf{1}_{t>0}\,dt$$

A straightforward calculation provides

$$\mu\left(T_{\pm n} \leq T_{-1}\right) \;=\; \int_{0\leq t_1 \leq t_2} dT_{\pm n}\left(\mu\right)(t_1)\; dT_{-1}\left(\mu\right)(t_2) \;\leq\; C(n^2+1)^{-1/2}$$

which gives the result.

6.3 Optimality of the decay rate

Theorem 4 shows that the boundary layer solution v converges *at least as* $y_2^{-1/2}$. One may wonder if this result is optimal, that is if we can find roughness distributions for which the speed of convergence is exactly given by $y_2^{-1/2}$. In other words, is the constant $\sigma_{(0,0)}$ of the theorem positive for some random distribution of roughness ? We have not so far been able to show optimality in this setting, but it can be established for the easier Dirichlet problem

$$\begin{cases} \Delta u = 0, & y_2 > 0 \\ \;\; u = \omega, & y_2 = 0 \end{cases}$$

where ω is a given boundary data. Although simpler, this system shares many features with the original system (1.4):

- If $\omega = \omega(y_1)$ is say L-periodic, the solution $u(y)$ converges exponentially fast to the constant $\alpha := L^{-1}\int_0^L \omega(y_1)dy_1$, as y_2 goes to infinity.
- If ω belongs as before to the probability space $(P, \mathcal{C}, \pi)$, one can show under assumption (H2) that

$$y_2\,\mathbb{E}|u(\omega,0,y_2)-\alpha|^2 \;\xrightarrow[y_2\to+\infty]{}\; \sigma^2 \;\geq 0, \quad \alpha := \mathbb{E}(\omega \mapsto \omega(0)).$$

Along the lines of [24, pages 21-22], we will exhibit a stationary measure π for which $\sigma > 0$. Of course, π is the law of the random process $\varphi(\omega, y_1) := \omega(y_1)$, so that we just need to characterize the random initial data. Let $G(\omega, y_1)$ a gaussian random process, of zero mean and covariance $\rho(z_1 - y_1)$, where $\rho \geq 0$ is a smooth even function with compact support. Note that such process exists: take $\rho = f * f$, with $f \geq 0$ an even smooth function with compact support. Then, its Fourier transform satisfies $\hat{\rho} = |\hat{f}|^2 \geq 0$, which ensures the required positivity property

$$\sum_{z_1,y_1} c(z_1)\,c(y_1)\rho(z_1-y_1) \;=:\; \int_{\mathbb{R}} |\sum_{z_1} c(z_1)e^{i\xi z_1}|^2 \hat{\rho}(\xi)\,d\xi \;\geq\; 0.$$

for any family c with compact support. Note moreover that this process defines almost surely smooth functions of y_1: indeed, a simple calculation yields

$$\mathbb{E}\left(\int_{[-R,R]} |\partial^k_{y_1} X(\cdot, y_1)|^2\, dy_1\right) \;=\; 2R\,(-1)^k\,\rho^{(2k)}(0) \;<\; \infty$$

so that $X(\omega,\cdot)$ is almost surely in the space $H^k_{loc}(\mathbb{R})$ and therefore smooth. Finally, we introduce

$$\varphi(\omega, y_1) \;=\; F(X(\omega, y_1))$$

for a smooth increasing function F with values in $(0,1)$. We stress that φ satisfies (H2) as ρ has compact support. We will show that the corresponding σ is positive. Suppose *a contrario* that $\sigma = 0$. For $y_2 \geq 1$, we introduce the measure π^{y_2} associated to the gaussian process with variance $\rho(z_1 - y_1)$ but mean

$$m(z_1, y_2) \;=\; \int \rho(z_1 - y_1) g(y_1, y_2)\, dy_1$$

where g will be given later. Note that π^{y_2} is associated to the random initial data

$$\varphi^{y_2}(\omega, y_1) \;:=\; F(X(\omega, y_1) + m(y_1, y_2)).$$

Standard computation yields

$$\begin{aligned} R(\omega, y_2) \;:=\; \frac{d\pi^{y_2}}{d\pi}(\omega) \;=\; \exp\Big(&\int g(y_1, y_2)\varphi(\omega, y_1) dy_1 \\ &- \frac{1}{2}\int\int \rho(z_1 - y_1) g(z_1, y_2)\, g(y_1, y_2) dz_1\, dy_1\Big). \end{aligned}$$

and

$$\int |R(\omega, y_2)|^2 d\pi = e^{H(y_2)}, \quad H(y_2) \;:=\; \int\int \rho(z_1 - y_1) g(z_1, y_2)\, g(y_1, y_2) dz_1\, dy_1.$$

If $\sigma = 0$, then a simple Cauchy-Schwartz inequality

$$\int \sqrt{y_2} |u(\omega, 0, y_2) - \alpha| d\pi^{y_2} \;\leq\; e^{\frac{1}{2}H(y_2)} \int y_2 |u(\omega, 0, y_2) - \alpha|^2 d\pi$$

and *goes to zero as $y_2 \to +\infty$ if H is bounded from above.*

Let u and v solutions associated to the initial data $\varphi(\omega, y_1)$ and $\varphi^{y_2}(\omega, y_1)$. As $m \geq 0$, by monotonicity, $v \geq u$. We can express u and v in terms of the Poisson Kernel, so that

$$(v - u)(\omega, 0, y_2) \;=\; \frac{2}{\pi}\int_{\mathbb{R}} \frac{y_2}{y_1^2 + y_2^2} \left(\varphi^{y_2}(\omega, y_1) - \varphi(\omega, y_1)\right) dy_1.$$

Now, we define for $y_2 \geq 1$

$$g(y_1, y_2) \;=\; \frac{1}{\sqrt{y_2}}\, G\left(\frac{y_1}{y_2}\right),$$

where $G \geq 0$ has compact support, $G = 1$ over $(-1, 1)$. On one hand, with this definition of g, one can check that

$$\sup_{y_2 \geq 1} H(y_2) \;=\; \sup_{y_2 \geq 1} y_2^{-1} \int\int \rho(z_1 - y_1) G\left(\frac{z_1}{y_2}\right)\, G\left(\frac{y_1}{y_2}\right) dz_1 dy_1 \;<\; +\infty$$

On the other hand, one has

$$\begin{aligned} &\int \sqrt{y_2}(v - u)(\omega, 0, y_2) d\pi \;\geq\; C \int_{\mathbb{R}} \left(\int F'(X(\omega, 0)) d\pi\right) \frac{y_2}{y_1^2 + y_2^2}\, \sqrt{y_2}\, m(y_1, y_2)\, dy_1 \\ &\geq\; C' \int_{\mathbb{R}} \left(\int F'(X(\omega, 0)) d\pi\right) \left(\inf_{y_2 \geq 1} \inf_{|y_1| \leq y_2} \int \rho(y_1 - s) G\left(\frac{s}{y_2}\right) ds\right) \left(\int_{-y_2}^{y_2} \frac{y_2}{y_1^2 + y_2^2} dy_1\right) \\ &\geq\; C'' \int_{\mathbb{R}} \rho(s) ds > 0. \end{aligned}$$

This implies that the quantity $\int \sqrt{y_2}\,(v(\omega, 0, y_2) - \alpha)\, d\pi$ does not go to zero, leading to a contradiction.

References

[1] Achdou, Y., Le Tallec, P., Valentin, F., and Pironneau, O. Constructing wall laws with domain decomposition or asymptotic expansion techniques. *Comput. Methods Appl. Mech. Engrg. 151*, 1-2 (1998), 215–232. Symposium on Advances in Computational Mechanics, Vol. 3 (Austin, TX, 1997).

[2] Achdou, Y., Mohammadi, B., Pironneau, O., and Valentin, F. Domain decomposition & wall laws. In *Recent developments in domain decomposition methods and flow problems (Kyoto, 1996; Anacapri, 1996)*, vol. 11 of *GAKUTO Internat. Ser. Math. Sci. Appl.* Gakkōtosho, Tokyo, 1998, pp. 1–14.

[3] Achdou, Y., Pironneau, O., and Valentin, F. Effective boundary conditions for laminar flows over periodic rough boundaries. *J. Comput. Phys. 147*, 1 (1998), 187–218.

[4] Amirat, Y., Bresch, D., Lemoine, J., and Simon, J. Effect of rugosity on a flow governed by stationary Navier-Stokes equations. *Quart. Appl. Math. 59*, 4 (2001), 769–785.

[5] Avellaneda, M., and Lin, F.-H. Compactness methods in the theory of homogenization. *Comm. Pure Appl. Math. 40*, 6 (1987), 803–847.

[6] Avellaneda, M., and Lin, F.-H. L^p bounds on singular integrals in homogenization. *Comm. Pure Appl. Math. 44*, 8-9 (1991), 897–910.

[7] Baladi, V. Decay of correlations. In *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, vol. 69 of *Proc. Sympos. Pure Math.* Amer. Math. Soc., Providence, RI, 2001, pp. 297–325.

[8] Basson, A., and Gérard-Varet, D. Wall laws for fluid flows at a boundary with random roughness. *Comm. Pure Applied Math., to appear* (2007). Available at `http://www.dma.ens.fr/~dgerardv/publications.html`.

[9] Bechert, D., and Bartenwerfer, M. The viscous flow on surfaces with longitudinal ribs. *J. Fluid Mech. 206*, 1 (1989), 105–129.

[10] Bresch, D., and Gérard-Varet, D. Roughness-induced effects on the quasi-geostrophic model. *Comm. Math. Phys. 253*, 1 (2005), 81–119.

[11] Bresch, D., and Milisic, V. Higher order boundary layer correctors and wall laws derivation: a unified approach. *Preprint arXiv:math/0611083 (2006).*

[12] De Bouard, A., Craig, W., Daz-Espinosa, O., Guyenne, P., and C., S. Long wave expansions for water waves over random topography. *Preprint arXiv:math/0506595* (2007).

[13] Durrett, R. *Probability: theory and examples*, second ed. Duxbury Press, Belmont, CA, 1996.

[14] Galdi, G. P. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, vol. 38 of *Springer Tracts in Natural Philosophy.* Springer-Verlag, New York, 1994. Linearized steady problems.

[15] Gérard-Varet, D. Highly rotating fluids in rough domains. *J. Math. Pures Appl. (9) 82*, 11 (2003), 1453–1498.

[16] Giaquinta, M. *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, vol. 105 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 1983.

[17] Giaquinta, M., and Modica, G. Nonlinear systems of the type of the stationary Navier-Stokes system. *J. Reine Angew. Math. 330* (1982), 173–214.

[18] Gilbarg, D., and Trudinger, N. S. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[19] Jäger, W., and Mikelić, A. On the roughness-induced effective boundary conditions for an incompressible viscous flow. *J. Differential Equations 170*, 1 (2001), 96–122.

[20] Jäger, W., and Mikelić, A. Couette flows over a rough boundary and drag reduction. *Comm. Math. Phys. 232*, 3 (2003), 429–455.

[21] Jäger, W., Mikelić, A., and Neuss, N. Asymptotic analysis of the laminar viscous flow over a porous bed. *SIAM J. Sci. Comput. 22*, 6 (2000), 2006–2028 (electronic).

[22] Jikov, V. V., Kozlov, S. M., and Oleĭnik, O. A. *Homogenization of differential operators and integral functionals*. Springer-Verlag, Berlin, 1994. Translated from the Russian by G. A. Yosifian [G. A. Iosif′yan].

[23] Luchini, P. Asymptotic analysis of laminar boundary-layer flow over finely grooved surfaces. *European J. Mech. B Fluids 14*, 2 (1995), 169–195.

[24] Varadhan, S., and Zygouras, N. Behavior of the solution of a random semi-linear heat equation. *Comm. Pure Applied Math., to appear* (2007). Available at `http://www-rcf.usc.edu/~zygouras/research.html`.

[25] Varadhan, S. R. S. *Stochastic processes*. Notes based on a course given at New York University during the year 1967/68. Courant Institute of Mathematical Sciences New York University, New York, 1968.

Acknowledgements

The author warmly thanks S.R.S Varadhan for pointing to reference [24], as well as Luis Silvestre and Thierry Levy for fruitful discussions.

Appendix: Measurability of $\tilde{w}$

We want to show here that

$$\tilde{w}(\omega, z) \;:=\; \int_{y_2=0} G_\omega(z,y)\, e_1\, dy, \quad z \text{ in } \Omega^{bl}(\omega), \quad \tilde{w}(\omega,z) := 0 \quad \text{otherwise}$$

defines a measurable function from P to $H^1_{loc}(\mathbb{R}^2)$. Let $0 \le \varphi_n \le 1$ a sequence of smooth functions with compact support, $\varphi_n|_{(-n,n)} = 1$. We define

$$w_n \;:=\; \int_{y_2=0} G_\omega(z,y)\, (\varphi_n e_1)\, dy, \quad z \text{ in } \Omega^{bl}(\omega), \quad w_n(\omega,z) := 0 \quad \text{otherwise.}$$

Note that w_n is the (unique) solution of

$$\begin{cases} -\Delta w_n + \nabla q_n = 0, \; x \in \Omega^{bl} \setminus \{y_2 = 0\}, \\ \text{div } w_n = 0, \; x \in \Omega^{bl}, \\ w_n|_{\partial\Omega^{bl}} \;=\; 0, \\ [w_n]|_{y_2=0} = 0, \quad [\partial_2 w_n - (0, q_n)]|_{y_2=0} = (-\varphi_n, 0), \end{cases} \tag{6.3}$$

satisfying $\int_{\Omega^{bl}(\omega)} |\nabla w_n|^2 \;<\; +\infty$. By the dominated convergence theorem applied to the integral formula, we get that $w_n \to \tilde{w}$ in L^2_{loc}. By the Cacciopoli inequality, the convergence is also true in H^1_{loc}. Thus, we just have to show measurability of w_n.

Let us define

$$V \;:=\; \left\{ v \in \dot{H}^1(\mathbb{R}^2), \quad \text{div } v = 0 \right\}, \quad V_\omega \;:=\; \left\{ v \in V, \quad v|_{\mathbb{R}^2 \setminus \Omega^{bl}(\omega)} = 0 \right\}.$$

Following the lines of [8, pages 15-16] it can be shown that the application $\omega \mapsto \pi(\omega)$, where $\pi(\omega) \in \mathcal{L}(V, V)$ is the orthogonal projection from V to V_ω, is measurable. Now, w_n is the unique fixed point of the contraction

$$w \mapsto \frac{1}{2}\pi(\omega)\,(w - v_n)$$

where v_n is the unique function of $H^1(\mathbb{R}^2)$ satisfying

$$\int_{\mathbb{R}^2} \nabla v_n \cdot \nabla \phi \;=\; 6 \int_{y_2=0} \varphi_n \, \phi_1 .$$

The measurability of w_n follows.